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# MATRIX..

Arrangement of nos in $m$ rows & $n$ columns is a matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

* Square matrix :- if $m=n$
* Zero matrix :- A sq. matrix whose all elements are zero
* Diagonal matrix :- A sq. matrix in which all diagonals elements are non-zero & other elements are 0/ non-zero.

  Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a diagonal matrix.

* Scalar matrix :- diagonal matrix whose all elements are same. Eg:- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

* Identity matrix :- Scalar matrix whose all diagonals elements are 1 & other elements are 0. Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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# ELEMENTARY TRANSFORMATIONS...

(i) $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$)
(ii) $R_i \to KR_i$ (or $C_i \to KC_j$) ($K \in \mathbb{R}$), $K \neq 0$
(iii) $R_i \to R_i + KR_j$ (or $C_i \to C_i + KC_j$)

# Inverse of a MATRIX

let A be any sq. matrix.
let B be any sq. matrix such that
$AB = I$ & $BA = I$ then B is inverse of A.

Q. Find the inverse of matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

by elementary row transformation

Ans. $A = I \cdot A$

$AA^{-1} = I$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + 5R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

** Elementary row transformation on the product of 2 matrices is equivalent to the same elementary row transformation on the pre-factor.

$$(AB)^* = A^*B$$
$$(AB)^{**} = AB^{**}$$

$R_3 \to \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

aminotes



$R_2 \to R_2 - 2R_3$ ; $R_1 \to R_1 - 3R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -15/2 & 11/2 & -3/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$I = A^{-1}A$

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# RANK OF A MATRIX...

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}_{3\times4}$$

Max. order of sub sq. matrix = 3

Determinant of the sub sq. matrix :- minor of the matrix

Rank → Order of non-zero largest minor.

* If all the minor of order 3 = 0, then rank is not defined
* Zero matrix has zero rank

Eg:- $\begin{bmatrix} & \\ & \end{bmatrix}_{4\times4}$ If the minor of order 4 is non-zero, then rank is 4. else we will check for 3, 2....

Q. Find the rank of matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$

Ans. let us consider $\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4 \neq 0$, ∵ this minor is non zero.

∴ Rank of $A = 2$

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Eg: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

# Rank of a matrix = no. of non-zero rows

#. Rank of a matrix

Let A be any $m \times n$ matrix, then it has sq. submatrices of order less than equal to min. of $m, n$ (whichever is less).

The determinant of that sq. submatrix is known as minor.

If all the minor of order $(r+1)$ are zero, but there is atleast 1 non-zero minor of order $r$, then rank of matrix is $r$ and is denoted by $\rho(A) = r$.

Note: (i) If A is a zero matrix/null matrix, then rank is zero.
(ii) If A is non-zero matrix, then rank of A is atleast 1.
(iii) If A is a matrix of order $m \times n$, then rank of A is less than min. of $(m, n)$, whichever is less.
** (iv) If I is identity matrix of order $n$, rank of I is $n$.
** (v) Rank remains unchanged by using any kind of elementary operation.

* In inverting method, both row & column transformations can be used.

Eg:- If a (3×4) matrix is given, then it is to be converted into :-

rank 3 :
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

rank 2 :-
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

They are known as normal form. The order of identity matrix gives the rank of matrix.

The normal form is written as $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$

Q. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$

Ans. $R_2 \to R_2 + 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 11 & 15 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$ ; $C_3 \to C_3 - 3C_1$ ; $C_4 \to C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 11 & 15 \end{bmatrix}$$

$C_2 \to 1/4\, C_2$ ; $C_3 \to 1/11\, C_3$ ; $C_4 \to 1/15\, C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$ ; $C_4 \to C_4 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= [I_2 | 0]$$

$\therefore \rho(A) = 2$ . (Ans.)

Q. Find the rank of $A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -3/2 \end{bmatrix}$

$R_1 \to R_1 - \frac{1}{2} R_2$ ; $R_3 \to R_3 + \frac{1}{4} R_2$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ 8 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 4 & 6 \end{bmatrix}$$

$\therefore \ \rho(A) = 1$ (No. of non-zero rows)

$C_1 \to C_2$

$$\begin{bmatrix} 4 & 8 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q. Find rank of $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

Ans. $R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 + R_1$ ; $R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$ ; $C_3 \to C_3 + 2C_1$ ; $C_4 \to C_4 - 3C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$R_3 \to \frac{R_3}{3}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= I_4$

$\therefore \rho(I) = 4$

Q. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

~~$R_1 \Rightarrow R_1 - R_2 + R_3$~~ $C_2 \to C_2 - 2C_1$ ; $C_3 \to C_3 - 3C_1$

Ans: $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -5 \\ 3 & -5 & -7 \end{bmatrix}$

$R_2 \to (-)R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 5 \\ 3 & -5 & -7 \end{bmatrix}$

~~$R_2 \Rightarrow R_2 + 2R_1$~~ $C_1 \to C_1 + 2C_2$ ; $C_3 \to C_3 - 5C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & -5 & 18 \end{bmatrix}$

$R_3 \to R_3 / 18$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7/18 & -5/18 & 1 \end{bmatrix}$

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$C_1 \to C_1 + \left(\frac{7}{18}\right)C_3$ ; $C_2 \to C_2 + \left(\frac{5}{18}\right)C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$\therefore \rho(A) = 3$. (Ans)

Q. Find rank of $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

$C_2 \to C_2 - C_1$ ; $C_3 \to C_3 + C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -2 & 3 & -1 \\ 3 & -2 & 3 & 1 \end{bmatrix}$$

~~$R_2 \to$~~ $C_2 \to C_2 + C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 3 & -1 \\ 3 & 1 & 3 & 1 \end{bmatrix}$$

$C_1 \to C_1 - C_2$ ; $C_3 \to$ $C_3 \to C_3 - 3C_2$

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$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$\Rightarrow I(2)$

$\therefore \rho(A) = 2$

Q. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$

Ans: $R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1 ; C_3 \to C_3 - 3C_1 ; C_4 \to C_4 - 4C_1 ; C_5 \to C_5 - 5C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & 3 & 4 \\ 1 & -1 & -2 & -3 & -4 \\ 1 & -1 & -2 & -3 & -4 \end{bmatrix}$$

$C_1 \to C_1 + C_2 ; C_3 \to C_3 - 2C_2 ; C_4 \to C_4 - 3C_2 ; C_5 \to C_5 - 4C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$

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# Row Echlon Form...

A matrix is said to be in row echlon form if

(i) The first non-zero entry in each non-zero row is 1.

(ii) The row containing only zeroes occur below all the non-zero rows.

(iii) The no. of zero before the first non-zero element in a row is less than, the no. of such zeroes in the next row.

NOTE:- (i) The no. of non-zero rows in row echlon form is the RANK of A MATRIX.

(ii) To convert in row echlon form we use only row operations.

Q. Convert in row-echlon form $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 7 & 4 & 5 \end{bmatrix}$

Ans. ~~$R_1 \to R_2 - 1$~~ $R_3 \to R_3 - (R_1 + R_2)$

$$\sim \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

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$$\sim \begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 7 \\ 3 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 2 & -22 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$$\sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -11 \\ 0 & 0 & 0 \end{bmatrix}$$

{All the (m) pts. are now satisfied}
{It is a triangular matrix}
all elements below diagonal elements are 0.

No. of non-zero rows : 2

$\therefore$ Rank of matrix : 2

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Q. Reduce the matrix $A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$

by row echlon form & hence find rank of A.

Ans. $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$

$R_3 \to R_3 - 2R_1$ ; $R_4 \to R_4 - 4R_1$

$= \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix}$

$R_3 \to R_3 + R_2$ ; $R_4 \to R_4 + R_2$

$= \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \to R_2 \div 11$

$$= \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -5/11 & 3/11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2.$

2) $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_4 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & -2 & -3 & -1 \end{bmatrix}$$

$R_3 \rightarrow \frac{1}{2} R_3$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_4 \rightarrow R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

# System Of Linear Equations...

$$a_{11}x_1 + a_{12}x_2 + \dots\dots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots a_{2n}x_2 = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots a_{mn}x_n = b_m$$

To solve these eqns. we use matrix below

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$Ax = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$ is known as coeffecient matrix.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

System of linear eqn

- Non-homogeneous system → $AX = B$
- Homogeneous system → $AX = 0$

# For non-homogeneous $(AX = B)$

(i) write augmented matrix as

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 4 & 5 & 6 & 12 \\ 7 & 8 & 9 & 13 \end{array}\right]$$

Eg:- $x + y = 2$ ; $2x + y = 3$

$$\approx \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 3 \end{array}\right]$$

* We can use row operations, but not column operations,

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Since solutions might be affected on using column operations.

(ii) Find rank of augmented matrix [ $\rho[A|B]$ ] and rank of A [ $\rho(A)$ ]

(iii) If both the ranks are equal, then the system has a solution & the system is consistent.

(iv) If they are not equal, the system does not possess any solution. The system is known as inconsistent.

The solutions may be unique/infinite.

Q. Find the soln. of the eqns.

$$x+y+z=11$$
$$2x-6y-z=0$$
$$3x+4y+2z=0$$

Ans.
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 0 \end{bmatrix}$$

$$A \quad X = B$$

For finding $\rho(A|B)$ ;

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -11 \\ 2 & -6 & -1 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 11 & 11 \\ 0 & -8 & -3 & -22 \\ 0 & 1 & -1 & -33 \end{array}\right]$$

$R_2 \longleftrightarrow R_3$

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$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & -1 & -33 \\ 0 & -8 & -3 & -22 \end{array}\right]$$

$R_3 \longrightarrow R_3 + 8R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & -1 & -33 \\ 0 & 0 & -11 & -286 \end{array}\right]$$

$R_3 \longrightarrow \left(\frac{-1}{11}\right) R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & -1 & -33 \\ 0 & 0 & 1 & 26 \end{array}\right]$$

$\therefore \rho(A) = 3$

Consider,

$$\begin{bmatrix} 1 & 1 & 11 \\ 0 & 1 & -33 \\ 0 & 0 & 26 \end{bmatrix}_{3\times3}$$

$\approx 26 \neq 0$

$\therefore \rho(A|B) = 3$

$\rho(A) = 3 \quad = \rho(A|B)$

$\therefore \; \rho(A) = \rho(A|B) = 3 = n$

$\Rightarrow$ The system has unique soln.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -33 \\ 26 \end{bmatrix}$$

$\Rightarrow x + y + z = 11$

$y - z = -33$

$z = 26$

$\Rightarrow y = -7 \quad , \quad x = -8$

So the soln. is

$x = -8, \; y = -7, \; z = 26$

9. Solve the system of linear eqns.

$x - 3y + z = -1$

$2x + y - 4z = -1$

$6x - 7y + 8z = 7$

Ans $[A|B] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 2 & 1 & -4 & -1 \\ 6 & -7 & 8 & 7 \end{array}\right]$

System of linear Eqn.

- Non-homogeneous $AX = B$
  - Consistent $\rho(A|B) = \rho(A)$
    - unique soln.
      - $\rho(A|B) = \rho(A) = r$ (no. of unknown variables)
    - Infinite soln.
      - $\rho(A|B) = \rho(A) < n$
      - $r < n$
      - Put $(n-r)$ as any arbitrary value say $K$.
  - Inconsistent $\rho(A|B) \neq \rho(A)$
    - the system does not have any soln.
- Homogeneous $[AX = 0]$ (always consistent)
  - Trivial
    - $\rho(A) =$ no. of variables
    - all the variables are 0.
  - infinitely many
    - $\rho(A) < n$
    - Put $(n-r)$ as any arbitrary value.

$R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 - 6R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 0 & 7 & -6 & 1 \\ 0 & 11 & 2 & 13 \end{array}\right]$$

$R_3 \longrightarrow R_3 - \frac{11}{7} R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 0 & 7 & -6 & 1 \\ 0 & 0 & 80/7 & 80/7 \end{array}\right]$$

$R_3 \longrightarrow \frac{7}{80} R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 0 & 7 & -6 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \text{—} \circledast$$

$\therefore \rho(A) = 3$

Cosider, $\begin{vmatrix} 1 & -3 & -1 \\ 0 & 7 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 7 \neq 0$

$\Rightarrow \rho(A|B) = 3$

Q3) $x_1 + x_2 - x_3 = 0$
$2x_1 - x_2 + x_3 = 3$
$4x_1 + 2x_2 - 2x_3 = 2$

Ans. $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & 2 \end{array}\right]$

$R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 - 4R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -2 & 2 & 2 \end{array}\right]$

$R_3 \to R_3 - \frac{2}{3} R_2$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore \rho(A) = 2$ ; $\rho(A|B) = 2$.

$\Rightarrow$ System is consistent, system has a solution.

$\because \rho(A) = \rho(A|B) = 2 < 3$ = no. of unknown variable

$\Rightarrow$ System has infinitely many soln.

** Q. For what values of $\lambda$ & $\mu$, this system has
(i) no solution.
(ii) unique soln.
(iii) infinite / more than 1 soln.

Ans.
$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

Ans.
$$\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$$

$$A=\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}$$ , if $|A|\neq 0$, $\rho(A)=3$

$R_2 \to R_2 - R_1$ ; $R_3 \to R_3 - R_1$

$$A\sim\begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 1 & \lambda-1\end{bmatrix}$$

$\therefore |A| = \lambda-1-2=\lambda-3$

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$\Rightarrow |A| = \lambda - 3$

If $\lambda \neq 3$, $\Rightarrow |A| \neq 0$.

$\therefore \rho(A) = 3$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

Case (i) If $\lambda \neq 3$, $\rho(A) = 3$

$\Rightarrow$ If $\mu \neq 10$, $\rho(A|B) = 3$

$\Rightarrow$ If $\mu = 10$, $\rho(A|B) = 3$

$\Rightarrow$ System has unique solution, where $\lambda \neq 3$, $\mu$ can have any value.

Case (ii) If $\lambda = 3$

$\Rightarrow |A| = 0$

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Put $n-r = 3-2 = 1$ variable as any arbitrary value

$\therefore$ ~~$x+y$~~ $\therefore x_1 + x_2 - x_3 = 0$

$-x_2 + x_3 = 1$

let $x_2 = k$

$\Rightarrow x_3 = 1+k$

$\therefore x_1 + k - 1 - k = 0$

$\Rightarrow x_1 = 1$

$\therefore$ The soln. of the eqn. is $x_1 = 1, x_2 = k, x_3 = 1+k$, where k can have any value & hence the system has $\infty$ solns.

Q1) $2x + y + 2z = 0$

$x + y + 3z = 0$

$4x + 3y + 9z = 0$

Q2) $x + 3y + 4z + 7w = 0$

$2x + 4y + 5z + 8w = 0$

$3x + y + 2z + 3w = 0$

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$\Rightarrow \rho(A) = 2$ $\left\{ \text{since } \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \right\}$

* If $\mu \neq 10$, $\rho(A|B) = 3$

$\Rightarrow \rho(A) \neq \rho(A|B)$

$\Rightarrow$ System has no solution, when $\lambda = 3, \mu \neq 10$

* If $\mu = 10$, $\rho(A|B) = 2$

$\Rightarrow \rho(A) = \rho(A|B) = 2 < 3 =$ no. of variables

$\Rightarrow$ System has infinitely many solutions, when $\lambda = 3, \mu = 10$

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Q. for what values of $b$, the system has (infinite) trivial soln. (non-trivial soln.)

$$2x+y+2z=0$$
$$x+y+3z=0$$
$$4x+3y+bz=0$$

Ans
$$A=\begin{bmatrix}2 & 1 & 2\\1 & 1 & 3\\4 & 3 & b\end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A=\begin{bmatrix}1 & 1 & 3\\2 & 1 & 2\\4 & 3 & b\end{bmatrix}$$

$R_2 \to R_2-2R_1$ ; $R_3 \to R_3-4R_1$

$$A=\begin{bmatrix}1 & 1 & 3\\0 & -1 & -4\\0 & -1 & b-12\end{bmatrix}$$

$R_3 \to R_3-R_2$

$$\sim\begin{bmatrix}1 & 1 & 3\\0 & -1 & -4\\0 & 0 & b-8\end{bmatrix}$$

(i) If $b \neq 8$, $|A| \neq 0$

$\Rightarrow \rho(A) = 3 = $ no. of variables

$\Rightarrow$ System has trivial soln. for $b \neq 8$

(ii) If $b = 8$, $|A| = 0$

Consider $\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$

$\Rightarrow \rho(A) = 2$

$\Rightarrow \rho(A) = 2 < 3 = $ no. of variables

$\Rightarrow$ System has non-trivial soln. for $b = 8$.

Q. $3x + 4y - z - 6w = 0$
$2x + 3y + 2z - 3w = 0$
$2x + y - 14z - 9w = 0$
$x + 3y + 13z + 3w = 0$

Ans $A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$

$R_4 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 - 2R_1$ ; $R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -12 \\ 0 & -5 & -30 & -18 \\ 0 & -2 & & \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & \cancel{-24}\ -40 & -15 \\ 0 & \cancel{-2}\ -5 & -40 & -15 \end{bmatrix}$$

$R_2 \to \dfrac{R_2}{-3}$ ; $R_3 \to \dfrac{R_3}{-5}$ ; $R_4 \to \dfrac{R_4}{-5}$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \end{bmatrix}$$

$R_4 \to R_4 - R_2$ ; $R_3 \to R_3 - R_2$

Case (i) If $\rho(A|B) = 3$, when $a+b+c \neq 0$
$\Rightarrow \rho(A) \neq \rho(A|B)$
$\therefore$ the system is inconsistent when $a+b+c \neq 0$

(ii) $\rho(A|B) = 2$, when $a+b+c = 0$
$\Rightarrow \rho(A) = \rho(A|B) = 2 < 3$ = no. of variables

$\therefore$ the system is consistent & has as many solns. when $a+b+c = 0$.

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = 2 < 4 =$ no. of variables

Put $n - r = 4 - 2 = 2$ variables as any arbitrary value.

$$x + 3y + 13z + 3w = 0 \quad \text{---} (1)$$
$$y + 8z + 3w = 0 \quad \text{---} (2)$$

let $y = k_1$ ; $w = k_2$

$\Rightarrow 8z = -y - 3w$

$= -k_1 - 3k_2$

$\Rightarrow \quad z = \frac{-1}{8}(k_1 + 3k_2)$

Putting in (1)

$$x + 3k_1 + 13\left[\frac{-1}{8}(k_1 + 3k_2)\right] + 3k_2 = 0$$

$$x = \frac{13k_1}{8} + \frac{39k_2}{8} - 3k_1 - 3k_2$$

$$= \frac{-11k_1 + 15k_2}{8}$$

Q.
$$-2x+y+z=a$$
$$x-2y+z=b$$
$$x+y-2z=c$$

Ans. $(A|B) = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right]$

$R_1 \leftrightarrow R_2$

$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \\ 1 & 1 & -2 & c \end{array}\right]$

$R_2 \to R_2 + 2R_1$ ; $R_3 \to R_3 - R_1$

$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ 0 & -3 & 3 & a+2b \\ 0 & 3 & -3 & c-b \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ 0 & -3 & 3 & a+2b \\ 0 & 0 & 0 & a+b+c \end{array}\right]$

$\Rightarrow \rho(A) = 2$

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Q. $x+2y-3z=1$
$2x-\lambda y-3z=2$
$x+2y+\lambda z=3$.
find the value of $\lambda$, for which system has unique soln. Also find the soln. when $\lambda=0$.

Ans. $$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & -\lambda & -3 & 2 \\ 1 & 2 & \lambda & 3 \end{array}\right]$$

For unique soln., $\rho(A) = 3$
$$\Rightarrow |A| \neq 0$$
$$\Rightarrow \begin{vmatrix} 1 & 2 & -3 \\ 2 & -\lambda & -3 \\ 1 & 2 & \lambda \end{vmatrix} \neq 0$$

$R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 - R_1$

$$\sim \begin{vmatrix} 1 & 2 & -3 \\ 0 & -\lambda-4 & 3 \\ 0 & 0 & \lambda+3 \end{vmatrix}$$

$$\Rightarrow -\{(\lambda+3)(\lambda+4)\} \neq 0$$

$\therefore$ For unique soln, $\lambda \neq -3$, $\lambda \neq -4$.

Methods to find soln. of system → Gauss elimination – triangular form
→ Gauss Jordan diagonal form

For $\lambda = 0$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 0 & -3 & 2 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & -4 & 3 & 0 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

~~ρ(A)~~ $R_3 \to \dfrac{R_3}{3}$ ; $R_2 \to \dfrac{R_2}{-4}$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$$

$R_2 \to R_2 + \dfrac{3}{4}R_3$ ; $R_1 \to R_1 + 3R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$$

$\therefore \cancel{\rho(A) = 3}$ $R_1 \to R_1 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$$

$\Rightarrow \rho(A) = 3$ & $\rho(A|B) = 3$

$\Rightarrow \rho(A) = \rho(A|B) = 3 =$ no. of variables

$\Rightarrow$ System has unique soln.

$x = 2, \; y = \frac{1}{2}, \; z = \frac{2}{3}$.

For $\lambda \neq -3, -4$ : system has unique soln.

Q. For what values of $\lambda$,

$x+y+z=1$

$x+2y+4z=\lambda$

$x+4y+10z=\lambda^2$

system has a soln. & find the soln.

Q. Show that the system

$$3x+4y+5z=a$$
$$4x+5y+6z=b$$
$$5x+6y+7z=c$$

, do not have a soln, unless ~~a+2b~~ a+c = 2b.

Ans

$$C(A|B) = \left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 1 & 1 & 1 & c-b \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 1 & 1 & 1 & b-a \\ 1 & 1 & 1 & c-b \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 1 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-2b+a \end{array}\right]$$

$\therefore \rho(A) = 2$

The system will have a soln. if

$\rho(A) = \rho(A|B) = 2$

$1 \Rightarrow \quad a+c-2b=0$

$\Rightarrow a+c = 2b$

Q. $x+y+z=1$

$2x+y+4z=K$

$4x+y+10z=K^2$

Ans. $\rho(A|B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & K \\ 4 & 1 & 10 & K^2 \end{array}\right]$

$R_2 \to R_2 - 2R_1 \; ; \; R_3 \to R_3 - 4R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & -3 & 6 & K^2-4 \end{array}\right]$

$R_3 \to R_3 - 3R_2$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & 0 & 0 & K^2-3K+2 \end{array}\right]$ $\quad K^2-4-3K+6$

For the system to be consistent,

$$\rho(A) = \rho(A|B)$$

$\because \rho(A) = 2$

$\therefore \rho(A|B) = 2$

$\therefore k^2 - 3k + 2 = 0$

$\Rightarrow k^2 - 2k - k + 2 = 0$

$\Rightarrow k(k-2) - 1(k-2) = 0$

$\Rightarrow (k-1)(k-2) = 0$

$\Rightarrow k = 1 \,;\, k = 2$

# Eigen Values & Eigen Vectors.

## # Characteristic equations.

Let a be any matrix of order n & I be any identity matrix of same order. Then the matrix $[A-\lambda I]$ is known as characteristics.

$$[A-\lambda I] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

~~If every row contain~~

The determinant of $A-\lambda I$ i.e. $|A-\lambda I|$ is polynomial in $\lambda$ with degree n is known as characteristic ~~eq~~ polynomial of A.

Roots of this eqn. are known as characteristic Roots, Eigen values | latent roots of the matrix A.

Eg- find the eigen values of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

→ The characteristic eqn. of A is

$$|A-\lambda I| = 0$$

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$$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$
$\Rightarrow 10 - 7\lambda + \lambda^2 - 4 = 0$
$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$
$\Rightarrow (\lambda-6)(\lambda-1) = 0$
$\Rightarrow \lambda = 1, 6.$ (Ans)

# Eigen Vectors

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a non-zero vector.

If $\lambda$ is an eigen value of A.
$\Rightarrow |A - \lambda I| = 0$

$\Rightarrow [A - \lambda I]$ is singular matrix
$\Rightarrow [A - \lambda I]X = 0$ has a solution (non-trivial soln.) — (*)
$\Rightarrow [A - \lambda I]X$ is also singular.

So an any non-zero vector X ~~with~~ which satisfies the eqn. (*) is known as eigen vector of X.

-- There are different eigen vectors for different eigen values. and it is always non-zero.

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$$-x_1+4x_2=0$$

let $x_2=k$

$$\Rightarrow x_1=4k$$

$X_2=\begin{bmatrix}4k\\k\end{bmatrix}=k\begin{bmatrix}4\\1\end{bmatrix}$ is an eigen vector corresponding to $\lambda=6$.

Q. $A=\begin{bmatrix}3&1&4\\0&2&6\\0&0&5\end{bmatrix}$.

Ans. $|A-\lambda I|=\begin{vmatrix}3-\lambda&1&4\\0&2-\lambda&6\\0&0&5-\lambda\end{vmatrix}$

$$=(3-\lambda)(2-\lambda)(5-\lambda)$$

The char. eqn. is

$$|A-\lambda I|=0$$

$$\Rightarrow(3-\lambda)(2-\lambda)(5-\lambda)=0$$

$$\Rightarrow\lambda=2,3,5$$

let $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be a non-zero vector corresponding to each value of $\lambda$.

Such that $(A-\lambda I)X=0$

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for $\lambda = 3$,
$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\sim \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{5}R_2$

$$\sim \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$10x_3 = 0 \quad \Rightarrow x_3 = 0$

$x_2 + 4x_3 = 0 \quad \Rightarrow x_2 = 0$

let $x_1 = k$

$\Rightarrow X_2 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 3$

Eg:- let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a non-zero vector. such that $(A - \lambda I) X = 0$.

Ans. $\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $\lambda = 1$, $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \rightarrow R_2 - \frac{1}{4} R_1$

$\sim \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4x_1 + 4x_2 = 0$
$x_1 + x_2 = 0$

let $x_2 = k$
$\Rightarrow x_1 = -k$

$\therefore X_1 = \begin{bmatrix} -k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 1$

For $\lambda = 6$ $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \rightarrow R_2 + R_1$

$\sim \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $\lambda = -2$ :-
$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$ ; $R_2 \longrightarrow R_2 - \frac{1}{3} R_1$

$$\sim \begin{bmatrix} 3 & 1 & 3 \\ 0 & 20/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{x_2 = \frac{20}{3}}$ $\quad \frac{20}{2} x_2 = 0 \quad \Rightarrow x_2 = 0$

$3x_1 + x_2 + 3x_3 = 0$

$x_1 + x_3 = 0$

let $x_1 = k$ ; $x_3 = -k$

$\Rightarrow X_1 = \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = -2$.

i.e. $\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda=2$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \longrightarrow R_3 - \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + 4x_3 = 0$$

$$6x_3 = 0$$

$$x_3 = 0$$

$\Rightarrow \; x_1 + x_2 = 0$

let $x_2 = k$

$\Rightarrow x_1 = -k$,

$\Rightarrow \; X_1 = \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 2$

Q. Find the eigen value $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Ans The char. eqn i

$$|A - \lambda I| = 0$$

$$\sim \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\sim (8-\lambda)\left[(7-\lambda)(3-\lambda) * -16\right] + 6\left[6(3-\lambda) + 8\right] + 2\left[24 - 2(7-\lambda)\right] = 0$$

$$\sim (8-\lambda)\left[5 + \lambda^2 - 10\lambda\right] + 6(26 - 6\lambda) + 2(10 + 2\lambda) = 0$$

$$\sim 40 + 8\lambda^2 - 80\lambda - 5\lambda - \lambda^3 + 10\lambda^2 + 156 - 36\lambda + 20 + 4\lambda = 0$$

$$\sim -\lambda^3 + 18\lambda^2$$

$$\sim (8-\lambda)\left[5 + \lambda^2 - 10\lambda\right] + 6(6\lambda - 10) + 2(10 + 2\lambda) = 0$$

$$\sim 8\lambda^2 - 80\lambda + 40 - \lambda^3 + 4\lambda^2 - 5\lambda - 40 + 40\lambda = 0$$

$$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Rightarrow -\lambda(\lambda^2 + 18\lambda + 45) = 0$$

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$$\sim \lambda(\lambda^2 - 15\lambda - 3\lambda + 45) = 0$$

$$\Rightarrow \lambda[\lambda(\lambda-15) - 3(\lambda-15)] = 0$$

$$\Rightarrow \lambda(\lambda-15)(\lambda-3) = 0$$

let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a non-zero vector, corresp. to each $\lambda$ such that

$$(A - \lambda I)X = 0 \quad \text{--- (4)}$$

for $\lambda = 0$

$$AX = 0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + 3R_1 \; ; \; R_3 \to R_3 - 4R_1$$

$$\sim \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_2$

$$\sim \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 - 4x_2 + 3x_3 = 0$

$-5x_2 + 5x_3 = 0$

let $x_3 = k$

$\Rightarrow x_2 = k$

$\Rightarrow x_1 = k/2$

$$\therefore X_1 = \begin{bmatrix} k/2 \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 3_3$,

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 \div (2)$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 - 5R_3$ ; $R_2 \to R_2 + 6R_3$

$$\sim \begin{bmatrix} 0 & 4 & 2 \\ 0 & -8 & -4 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{2} R_2$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & -8 & -4 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 = 0$$
$$-8x_2 - 4x_3 = 0$$

$$x_2 = k$$
$$x_3 = -2k$$
$$x_1 = 2k$$

$$X_2 = \begin{bmatrix} 2k \\ k \\ -2k \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

for $\lambda = 5$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 \div 2$

$$\sim \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 1 & -2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 + 7R_3$ ; $R_2 \to R_2 + 6R_3$

$$\sim \begin{bmatrix} 0 & -20 & -40 \\ 0 & -20 & -40 \\ 1 & -2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-10x_2 - 20x_3 = 0$$

$$x_1 - 2x_2 - 6x_3 = 0$$

let $x_3 = k$

$x_2 = -2k$

$x_1 = 2k$

$$X_3 = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

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** Q. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The char. eqn. is

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$$

$\sim \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0$

$\lambda = -3, -3, 5$

let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore X_1 = k_1\begin{bmatrix}-2\\1\\0\end{bmatrix}$ & $X_2 = k_2\begin{bmatrix}3\\0\\1\end{bmatrix}$

For $\lambda = 5$

$$\begin{bmatrix}-7 & 2 & -3\\2 & -4 & -6\\-1 & -2 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3 \to (-)R_3$

$$\sim\begin{bmatrix}-7 & 2 & -3\\2 & -4 & -6\\1 & 2 & 5\end{bmatrix}$$

$R_2 \to R_2 - 2R_3 \; ; \; R_1 \to R_1 + 7R_3$

$$\sim\begin{bmatrix}0 & 16 & 32\\0 & -8 & -16\\1 & 2 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_1 \to R_1 + (2)R_2$

$$\sim\begin{bmatrix}0 & 0 & 0\\0 & -8 & -16\\1 & 2 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

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For $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + 2x_2 - 3x_3 = 0$

Let $x_1 = k_1$ ; $x_2 = k_2$

$\therefore x_3 = 3k_2 - 2k_1$

$$\therefore X_1 = \begin{bmatrix} -2k_1 \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3k_2 \\ 0 \\ k_2 \end{bmatrix}$$

$$= \underbrace{k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{x_1} + \underbrace{k_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}_{x_2}$$

$\because$ roots are equal, therefore, linearly independent

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$$x_1 + 2x_2 + 5x_3 = 0$$

$$x_2 + 2x_3 = 0$$

$$x_3 = k \quad ; \quad x_2 = -2k$$

$$x_1 = -k$$

$$\therefore X_3 = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Date 30/8/16

# CAYLAY HAMILTON THEOREM

** It states that every sq. matrix satisfies its characteristic eqns.

Eg:- $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ verify Caylay Hamilton for the same.

Ans. The char. eqn. of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(5-\lambda) - 9 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 1 = 0$$

Caylay Hamilton states that if $\lambda$ is replaced by a , it will satisfy the characteristic eqn.

we will show that

$$A^2 - 7A + I = 0$$

$$A^2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix}$$

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$$LHS = A^2 - 7A + I$$

$$= \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix} - 7\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \therefore LHS = RHS$$

$\therefore$ Cayley Hamilton method is verified

# To find inverse of (A)

$$A^2 - 7A + I = 0$$

Premultiply by $A^{-1}$

$$A - 7I + A^{-1} = 0$$

$$\Rightarrow A^{-1} = 7I - A$$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ find characteristic eqn. of A.

Ans. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$

$\Rightarrow (2-\lambda)\left[(2-\lambda)^2 - 1\right] + 1\left[-2+\lambda+1\right] + 1\left[1 - 2 + \lambda\right]$

$\Rightarrow (2-\lambda)\left[3 + \lambda^2 - 4\lambda\right] + 1\left[\lambda - 1\right] + 1\left[\lambda - 1\right]$

$\Rightarrow 6 + 2\lambda^2 - 8\lambda - 3\lambda - \lambda^3 + 4\lambda^2 + 2\lambda - 2$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

We will show that
$A^3 - 6A^2 + 9A - 4 = 0$

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$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\text{LHS} = A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$$

$$+ \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} + \begin{bmatrix} 9 & -1 & 1 \\ -1 & 9 & -1 \\ 1 & -1 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Cayley Hamilton thm. is verified.

$$A^3 - 6A^2 + 9A - 4I = 0$$

Premultiply by $A^{-1}$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$4A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

$$= \frac{1}{4}\left\{ \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \right\}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 3/4 & 1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 \\ -1/4 & 1/4 & 3/4 \end{bmatrix}$$

*If a & b are in diff rows, then determinant method would be used!

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Q. $x + ay + z = 3$
$x + 2y + 2z = b$
$x + 5y + 3z = 9$

Ans. $A = \begin{bmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 \\ b \\ 9 \end{bmatrix}$

For unique soln.,
$\rho(A) = 3$
$\Rightarrow |A| \neq 0$

$\Rightarrow \begin{vmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{vmatrix} \neq 0$

$a \neq -1$

$\therefore$ The system has unique soln. when $a \neq -1$ & $b$ may have any value.

Put $a = -1$
$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{bmatrix}$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 1 & 2 & 2 & b \\ 1 & 5 & 3 & 9 \end{array}\right]$$

$R_2 \to R_2 - R_1$ ; $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & b-3 \\ 0 & 6 & 2 & 6 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 6 & 2 & 6 \\ 0 & 3 & 1 & b-3 \end{array}\right]$$

$R_3 \to R_3 - \frac{1}{2}R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 6 & 2 & 6 \\ 0 & 0 & 0 & b-6 \end{array}\right]$$

$\rho(A) = 2$

Case(i) If $b \neq 6$ $\Rightarrow \rho(A|B) = 3$

$\Rightarrow \rho(A) \neq \rho(A|B)$

The system has no. soln. when $a=1$, $b\neq 6$

Case (ii) If $b=6$, $\rho(A|B)=2$

$\Rightarrow \rho(A)=\rho(A|B) <$ no. of variables

~~ooth~~ ∴ The system has $\infty$ soln.
where $a=1, b=6$

# PROPERTIES OF EIGEN VECTORS.

1) If $\lambda$ is an eigen value of A, then $\lambda$ is also eigen value of $A^T$ i.e. A & A' have same eigen value.

PROOF:- Since $\lambda$ is an eigen value of A.

then $|A - \lambda I| = 0$

$\Rightarrow |(A-\lambda I)'| = 0$ (as $|A| = A^T$)

$\Rightarrow |A' - (\lambda I)'| = 0$ [as $(A \pm B)' = A' \pm B'$]

$\Rightarrow |A' - \lambda I| = 0$

$\Rightarrow \lambda$ is also an eigen value of A

** 2) If $\lambda$ is an eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of $A^{-1}$.

PROOF :- Since $\lambda$ is an eigen value of A

the $\exists$ (there exists) a non-zero vector X such that

$(A - \lambda I) X = 0$

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$\Rightarrow AX - \lambda(IX) = 0$

$\Rightarrow AX = \lambda X$ —— (*)

We will show that,

$$\boxed{A^{-1}X = \frac{1}{\lambda}X}$$

Premultiply (*) by $A^{-1}$, we get

$A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow (A^{-1}A)X = \lambda(A^{-1}X)$

$\Rightarrow IX = \lambda(A^{-1}X)$

$\Rightarrow \frac{1}{\lambda}X = A^{-1}X$

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of $A^{-1}$.

3) If $\lambda$ is an eigen value of $A$, then $k\lambda$ is an eigen value of $kA$.

Proof continued from (1) of 2, multiply (*) by k

$(KA)X = (K\lambda)X$,

4) If $\lambda$ is an eigen value of $A$, then $\lambda^n$ is an eigen value of $A^n$, where $n$ is positive integer.

Proof :- Since $\lambda$ is an eigen value of $A$, then there exists a non-zero vector $X$ such that

$$(A-\lambda I)X = 0$$

$\Rightarrow AX - \lambda(IX) = 0$

$\Rightarrow AX = \lambda X$ ——— (*)

Premultiply (*) by $A$, we get

$A(AX) = A(\lambda X)$

$\Rightarrow A^2\lambda = \lambda(AX)$

$\Rightarrow A^2X = \lambda(\lambda X)$ (by using (*))

$\Rightarrow \boxed{A^2X = \lambda^2 X,}$ ——— (**)

Again, premultiply (**) by A, we get

$$A(A^2X) = A(\lambda^2 X)$$

$$\Rightarrow A^3X = \lambda^2(AX)$$

$$\Rightarrow A^3X = \lambda^2(\lambda X) \quad \text{(by using (*))}$$

$$\Rightarrow A^3X = \lambda^3 X$$

⋮

$$\boxed{A^n X = \lambda^n X}$$

$\Rightarrow \lambda^n$ is an eigen value of $A^n$

NOTE:- So, if $\lambda_1, \lambda_2 \ldots \lambda_n$ are eigen value of A, then $\lambda_1^n, \lambda_2^n \ldots \lambda_m^n$ are eigen value of $A^n$.

** 5) Sum of eigen values is equal to the sum of the principle diagonal elements.

** 6) Product of eigen values is equal to the determinant of A.

Eg:- $A = \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 2 & 2 & 2 & -3 \\ 1 & 1 & -4 & -4 \end{bmatrix}$

Sum of eigen values $= 1+1+2-4 = 0$

** 7) Eigen value of triangular matrix are just the diagonal elements

Eg:- $A = \begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

Eigen values are $1, 1, 2, -4$

8) Eigen values of diagonal matrix, are just its diagonal elements

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** 9) If $\lambda$ is an eigen value of non-singular matrix A, then show that $\frac{|A|}{\lambda}$ is an eigen value of (adj A)

PROOF:- ~~Since $\lambda$ is an eigen value of A, then $\lambda^n$ is an eigen value of $A^n$, where~~

Since $\lambda$ is an eigen value of A, there exists a non-zero vector $\lambda$ such that

$$(A - \lambda I)X = 0$$

$$\Rightarrow AX - \lambda I X = 0$$

$$\Rightarrow AX = \lambda \alpha \quad \text{———} \quad (*)$$

Premultiplying by (adj. A)

$$\Rightarrow (\text{adj. } A) AX = \lambda (\text{adj } A) X$$

$$A^{-1} = \frac{(\text{adj. } A)}{|A|}$$

$$\Rightarrow \frac{1}{A} = \frac{(\text{adj. } A)}{|A|}$$

$$\Rightarrow |A| = (\text{Adj. } A) A$$

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$\Rightarrow |A| X = \lambda (adj. A) X$

$\Rightarrow \frac{|A|}{\lambda} X = (adj. A) X$

$$\Rightarrow \boxed{\frac{|A|}{\lambda} = adj. A}$$

Q. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ~~Eigen val~~

Eigen value of A are 2, 2, 2

$\Rightarrow$ Eigen values of $A^8$ are $2^8, 2^8, 2^8$

Q. $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ find adj.(A) by using Cayley Hamilton law, find $A^8$

$|A - \lambda I| = 0$

$\Rightarrow (1-\lambda)(-1-\lambda) - 4 = 0$

$\Rightarrow -(1-\lambda^2) - 4 = 0$

$\Rightarrow \lambda^2 - 5 = 0$

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$\therefore$ By Cayley Hamilton thm,
$$A^2 - 5I = 0$$

Premultiply by $A^{-1}$
$$A - 5A^{-1} = 0$$
$$A^{-1} = \frac{1}{5}[A]$$

$$= \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$

$$|A| = -5$$

$$(adj.\ A) = \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$$

We have,
$$A^2 - 5I = 0$$
$$A^2 = 5I$$
$$\Rightarrow (A^2)^4 = 5^4 I^4$$
$$\Rightarrow A^8 = 625I$$

$$\Rightarrow A^8 = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$$

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# DIAGONALIZATION OF A MATRIX

## # Similar Matrix

Let A be any sq. matrix & B be any sq. matrix of same order, the B is said to be similar to A, if there exists a non-singular matrix P such that

$$A = P^{-1} B P$$

or

$$PA = BP$$

$$\Rightarrow B = P A P^{-1}$$

$P = [x_1 \quad x_2 \quad x_3]$ is known as modal matrix

Let A be any sq. matrix, then we will find a non-singular matrix P such that

$$P^{-1} A P = D \quad , \text{where } D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$. Reduce into diagonal form

Ans. The char. eqn. of A is

$$|A - \lambda I| = 0$$

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$$\sim \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-2\right] - 1\left[2-2(2-\lambda)\right] = 0$

$\Rightarrow (1-\lambda)(4-5\lambda+\lambda^2) - 1(2\lambda-2) = 0$

$\Rightarrow 4-5\lambda+\lambda^2-4\lambda+5\lambda^2-\lambda^3-2\lambda+2=0$

$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0$

$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6=0$

$$\begin{array}{r|l}
 & \lambda^2-5\lambda+6 \\
\lambda-1 & \lambda^3-6\lambda^2+11\lambda-6 \\
 & \lambda^3-\lambda^2 \\
 & (-) \\ \hline
 & -5\lambda^2+11\lambda-6 \\
 & -5\lambda^2+5\lambda \\
 & (-) \\ \hline
 & 6\lambda-6
\end{array}$$

$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3)=0$

$\therefore \lambda = 1, 2, 3$ are the eigen values of A

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let $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be an eigenvector of A corresp. to each $\lambda$.

$$[A-\lambda I]X=0 \quad \text{— (1)}$$

$\lambda=1$; $(A-I)X=0$

$$\Rightarrow \begin{bmatrix}0&0&-1\\1&1&1\\2&2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$x_3=0$$
$$x_1+x_2+x_3=0$$

let $x_2=k$
$x_1=-k$.

$\therefore X_1=\begin{bmatrix}-k\\k\\0\end{bmatrix}=k\begin{bmatrix}-1\\1\\0\end{bmatrix}$ is eigenvector corresp. to $\lambda=1$

For $\lambda=2$; $\begin{bmatrix}-1&0&-1\\1&0&1\\2&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$x_1+x_3=0$$

let $x_1=k$ ; $x_3=-k$

$$\sim \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$$

$\Rightarrow$ $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$$

$R_2 \to R_2 + R_1$ ; $R_3 \to \frac{1}{2} R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} P$$

$R_1 \to R_1 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1/2 \\ 1 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} P$$

$R_2 \to R_2 (-1) R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix} P$$

~~Goes~~

$$\therefore P^{-1} = \begin{bmatrix} 0 & 1 & -1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$$

Consider

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

show this

# To find higher powers of A

# use of diagnolization :- we can find any power of A.

$$P^{-1}AP = D$$

$$\Rightarrow AP = PD$$

$$\Rightarrow A = PDP^{-1}$$

Now, $A^2 = A \cdot A = (PDP^{-1})(PDP^{-1})$

$$= PD(P^{-1}P)DP^{-1}$$

$$= PD^2P^{-1}$$

$A^3 = A^2 \cdot A = (PD^2P^{-1}) \cdot (PDP^{-1})$

$$= PD^2(P \cdot P)DP^{-1}$$

$$= PD^2P^{-1}$$

$$2x_1 + 2x_2 + x_3 = 0$$

$$-k = 2x_2$$

$$\Rightarrow x_2 = \frac{-k}{2}$$

$$\therefore X_2 = \begin{bmatrix} k \\ -k/2 \\ -k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1/2 \\ -1 \end{bmatrix} = k\begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix}$$ is an eigenvector corresp. to $\lambda = 2$

for $\lambda = 3$,

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 2x_2 = 0$$

Let $x_2 = k$ ; $x_1 = -k$

$$-2x_1 - x_3 = 0$$

$$2k = x_3$$

$$\therefore X_3 = \begin{bmatrix} -k \\ k \\ 2k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$ is an eigenvector corresp. to $\lambda = 2$

Let $P = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

# k can have any value.

$$\boxed{A^n = PD^nP^{-1}}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

# MODULE-2 (MULTI-VARIABLE CALCULUS)

$y = f(x)$

$z = f(x, y)$ represents a surface in 2-dimensions & $x$ and $y$ both are independent variables, $z$ is dependent on $x/y$

$u = f(x, y, z)$

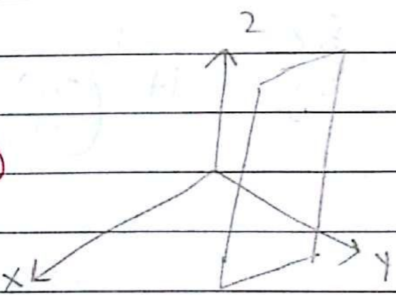
$z = f(x_1, x_2 \ldots\ldots x_n)$

# Derivatives of $z = f(x, y)$

Diff. $z = f(x, y)$ w.r.t. $x$

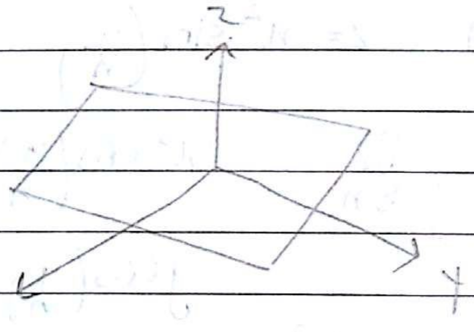
δ → dell
↓
rtial
fferentiation

$\frac{\delta z}{\delta x}$ (y remain constant)




Diff. $z = f(x, y)$ w.r.t. $y$

$\frac{\delta z}{\delta y}$ (x remain constant)




Eg:- $z = x^2 y$

$\frac{\delta z}{\delta x} = 2xy$ $\qquad$ $\frac{\delta z}{\delta y} = x^2$

Q. Find first order partial derivatives of the following:-

(i) $z = \log(x^2+y^2)$

$$\frac{\delta z}{\delta x} = \frac{(2x)}{(x^2+y^2)} \qquad \frac{\delta z}{\delta y} = \frac{2y}{x^2+y^2}$$

(ii) $z = \tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\delta z}{\delta x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) = \frac{y^2}{x^2+y^2(y)} = \frac{y}{x^2+y^2}$$

$$\frac{\delta z}{\delta y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right) = \frac{-x}{x^2+y^2}$$

(iii) $z = x^2 \sin\left(\frac{y}{x}\right)$

$$\frac{\delta z}{\delta x} = x^2\cos\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + \sin\left(\frac{y}{x}\right)2x$$

$$= -y\cos\left(\frac{y}{x}\right) + 2x\sin\left(\frac{y}{x}\right)$$

$$\frac{\delta z}{\delta y} = x^2\cos\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x\cos\left(\frac{y}{x}\right)$$

# Second Order Partial Derivative ...

$$z = f(x, y)$$

$\frac{\delta z}{\delta x}$ & $\frac{\delta z}{\delta y}$ have both $x$ & $y$ and thus act as functions with 2 variables..

$\frac{\delta z}{\delta x} = g(x, y)$ $\qquad$ $\frac{\delta z}{\delta y} = k(x, y)$

Diff. w.r.t. $x$

$$\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta x}\right) = \frac{\delta^2 z}{\delta x^2} \text{ --- (1)}$$

Diff w.r.t $x$

$$\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta y}\right) = \frac{\delta^2 z}{\delta x \delta y} \text{ --- (1)}$$

Diff. w.r.t $y$

$$\frac{\delta}{\delta y}\left(\frac{\delta z}{\delta x}\right) = \frac{\delta^2 z}{\delta y \delta x} \text{ --- (2)}$$

Diff w.r.t $y$

$$\frac{\delta}{\delta y}\left(\frac{\delta z}{\delta y}\right) = \frac{\delta^2 z}{\delta y^2} \text{ --- (2)}$$

Where (1) & (2) are always equal

$$\frac{\delta^2 z}{\delta x^2} = \frac{\delta^2 z}{\delta y^2} = \frac{\delta^2 z}{\delta x \delta y}$$

Q1) Show that, $\frac{x \delta^2 u}{\delta y \delta z} = \frac{y \delta^2 u}{\delta z \delta x} = \frac{z \delta^2 u}{\delta x \delta y}$.

Where $u = \log(x^2 + y^2 + z^2)$

$$\frac{\delta u}{\delta x} = \frac{1}{x^2+y^2+z^2}(2x) = \frac{2x}{x^2+y^2+z^2} \quad \text{--- } (1)$$

$$\frac{\delta u}{\delta y} = \frac{2y}{x^2+y^2+z^2} \quad \text{--- } (3)$$

$$\frac{\delta u}{\delta z} = \frac{2z}{x^2+y^2+z^2} \quad \text{--- } (4)$$

Diff. (4) wrt y

$$\frac{\delta u^2}{\delta y \delta z} = 2z\left(\frac{-2y}{(x^2+y^2+z^2)^2}\right)$$

$$= -\frac{4zy}{(x^2+y^2+z^2)^2}$$

$$\therefore x\frac{\delta u^2}{\delta y \delta z} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{--- } (A)$$

Diff. (3) wrt, x

$$\frac{\delta^2 u}{\delta y \delta x} = 2y\left(\frac{-2x}{(x^2+y^2+z^2)^2}\right)$$

$$z\frac{\delta^2 u}{\delta x \delta y} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{--- } (B)$$

Diff. (2) wrt z

$$\frac{\delta^2 u}{\delta x \delta z} = 2x\left(\frac{-2z}{(x^2+y^2+z^2)^2}\right)$$

$$\frac{\delta u}{\delta x} = \frac{1}{x^2+y^2+z^2}(2x) = \frac{2x}{x^2+y^2+z^2} \quad \text{--- (2)}$$

$$\frac{\delta u}{\delta y} = \frac{2y}{x^2+y^2+z^2} \quad \text{--- (3)}$$

$$\frac{\delta u}{\delta z} = \frac{2z}{x^2+y^2+z^2} \quad \text{--- (4)}$$

Diff. (4) wrt y

$$\frac{\delta u^2}{\delta y \delta z} = 2z\left(\frac{-2y}{(x^2+y^2+z^2)^2}\right)$$

$$= -\frac{4zy}{(x^2+y^2+z^2)^2}$$

$$\therefore x\frac{\delta u^2}{\delta y \delta z} = -\frac{4xyz}{(x^2+y^2+z^2)^2} \quad \text{--- (A)}$$

Diff. (3) wrt, x

$$\frac{\delta^2 u}{\delta y \delta x} = 2y\left(\frac{-2x}{(x^2+y^2+z^2)^2}\right)$$

$$z\frac{\delta^2 u}{\delta x \delta y} = -\frac{4xyz}{(x^2+y^2+z^2)^2} \quad \text{--- (B)}$$

Diff. (2) wrt z

$$\frac{\delta^2 u}{\delta x \delta z} = 2x\left(\frac{-2z}{(x^2+y^2+z^2)^2}\right)$$

$$\frac{\delta u}{\delta x} = \frac{1}{x^2+y^2+z^2}(2x) = \frac{2x}{x^2+y^2+z^2} \quad \text{—— (2)}$$

$$\frac{\delta u}{\delta y} = \frac{2y}{x^2+y^2+z^2} \quad \text{—— (3)}$$

$$\frac{\delta u}{\delta z} = \frac{2z}{x^2+y^2+z^2} \quad \text{—— (4)}$$

Diff. (4) wrt y

$$\frac{\delta u^2}{\delta y \delta z} = 2z\left(\frac{-2y}{(x^2+y^2+z^2)^2}\right)$$

$$= -\frac{4zy}{(x^2+y^2+z^2)^2}$$

$$\therefore x\frac{\delta u^2}{\delta y \delta z} = -\frac{4xyz}{(x^2+y^2+z^2)^2} \quad \text{—— (A)}$$

Diff. (3) wrt, x

$$\frac{\delta^2 u}{\delta y \delta x} = 2y\left(\frac{-2x}{(x^2+y^2+z^2)^2}\right)$$

$$z\frac{\delta^2 u}{\delta x \delta y} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{—— (B)}$$

Diff. (2) wrt z

$$\frac{\delta^2 u}{\delta x \delta z} = 2x\left(\frac{-2z}{(x^2+y^2+z^2)^2}\right)$$

$$\frac{y\,\delta^2 u}{\delta z\,\delta x} = -\frac{4xyz}{(x^2+y^2+z^2)^2} \quad \text{—— (C)}$$

$\therefore$ From (A), (B) & (C)

$$\frac{x\,\delta^2 u}{\delta y\,\delta z} = \frac{y\,\delta^2 u}{\delta x\,\delta z} = \frac{z\,\delta^2 u}{\delta x\,\delta y}$$

Hence proved

Q2) $u = x^y$

Prove :- $\frac{\delta^3 u}{\delta x^2 \delta y} = \frac{\delta^3 u}{\delta x\,\delta y\,\delta x}$

Ans. $u = x^y$ —— (1)

$$\frac{\delta u}{\delta x} = y\,x^{y-1} \quad \text{—— (2)}$$

$$\frac{\delta u}{\delta y} = x^y \log x$$

Q3) $u = \log(x^3+y^3+z^3-3xyz)$

Show that

$$\left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Ans. $$\frac{\delta u}{\delta x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\delta u}{\delta y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

$$\frac{\delta u}{\delta z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

$$\therefore \frac{\delta u}{\delta x}+\frac{\delta u}{\delta y}+\frac{\delta u}{\delta z} = \left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)u$$

$$= \left(\frac{3[x^2+y^2+z^2-xy-yz-zx]}{x^3+y^3+z^3-3xyz}\right)$$

$$= \frac{3}{x+y+z}$$

$$LHS = \left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\delta}{\delta x}\left(\frac{3}{x+y+z}\right)+\frac{\delta}{\delta y}\left(\frac{3}{x+y+z}\right)+\frac{\delta}{\delta z}\left(\frac{3}{x+y+z}\right)$$

$$= 3\left[\frac{-1 \quad -1 \quad -1}{(x+y+z)^2}\right] = \frac{-9}{(x+y+z)^2} = RHS$$

Hence prove

$\{ r' = |r'| , r' = xi + yj \}$



* Q. If $u = f(r)$, $r^2 = x^2 + y^2$, Hence show that

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{1}{r} f'(r) + f''(r)$$

Ans. $u$ is a function of $r$, $r$ is a function of $x$ & $y$.

$\therefore$ $u$ is a function of $x$ & $y$

$r = g(x, y)$ $\quad u = f(r)$

$u = f(g(x, y))$

Since $u = f(r)$ —— ①

Diff. w.r.t $x$

$$\frac{\delta u}{\delta x} = f'(r) \cdot \frac{\delta r}{\delta x} \quad —— ②$$

Diff. w.r.t $y$ —— ③

$$\frac{\delta u}{\delta y} = f'(r) \frac{\delta r}{\delta y}$$

$r^2 = x^2 + y^2$

Diff. w.r.t $x$ :-

$\cancel{\frac{\delta r \delta r}{\delta x} = 2x}$ $\quad 2r \frac{\delta r}{\delta x} = 2x$

$\cancel{\frac{\delta r}{\delta x} = \frac{x}{r}} \Rightarrow \frac{\delta r}{\delta x} = \frac{x}{r}$

Similarly, $\frac{\delta r}{\delta y} = \frac{y}{r}$

Substituting in ② & ③

$\Rightarrow \frac{\delta u}{\delta x} = \frac{x}{r} f'(r)$ —④ ; $\frac{\delta u}{\delta y} = \frac{y}{r} f'(r)$ —⑤

Diff. ④ w.r.t. $x$

$\frac{\delta^2 u}{\delta x^2} = \cancel{\frac{1}{r}\left\{ f'(r) + x f \right.}$

$= \frac{1}{r} f'(r) + \frac{x}{r} f'(r) \frac{\delta r}{\delta x} - \frac{x}{r^2}\left(\frac{\delta r}{\delta x}\right) f''(r)$

# Euler's Theorem For Homogeneous Eqns. Of X & Y...

$$a_n x^n + a_{n-1} x^{n-1} y + a_{n-2} x^{n-2} y^2 + \ldots \ldots a_0 y^n = 0$$

$$\Rightarrow x^n \left[ a_n + a_{n-1}\left(\frac{y}{x}\right) + a_{n-2}\left(\frac{y}{x}\right)^2 + \ldots \ldots a_0 \left(\frac{y}{x}\right)^n \right] = 0$$

$$\Rightarrow x^n \phi(y/x) = 0$$

$$\therefore f(x,y) = x^n \underbrace{\phi}(y/x) \quad \text{or} \quad y^n \phi(x/y)$$

represents polynomial interms of $x$ & $y$

Euler theorem states that $z = f(x,y)$ is a homogeneous function of degree $n$, then

$$x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = nz$$

PROOF :- Let $z = f(x,y)$ be a homogeneous function of degree $n$.

$$z = f(x,y) = x^n \phi(y/x)$$

Diff. w.r.t $x$

$$\frac{\delta z}{\delta x} = n x^{n-1}\phi\left(\frac{y}{x}\right) + x^{n}\phi'\left(\frac{-y}{x^{2}}\right)\left(\frac{y}{x}\right)$$

$$\frac{\delta z}{\delta y} = \cancel{nx^{n-1}\phi\left(\frac{y}{x}\right)} + x^{n}\phi'\left(\frac{1}{x}\right)\left(\frac{y}{x}\right)$$

$$\therefore \text{LHS} = x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y}$$

$$= n x^{n}\phi\left(\frac{y}{x}\right) - x^{n-2}(y^{2})\phi' + x^{n-2}(y^{2})\phi'$$

$$= n\left\{x^{n}\phi(y/x)\right\}$$

$$= nz$$

Q1) $z = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)$. Prove:- $x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = 0$

Ans. $z = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$= x^{0}\left[\sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$Z = x^{0}\phi(y/x)$$

$\therefore$ $x$ is a homogeneous function of $Z$

$\therefore$ By Euler's theorem,

$$x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = 0\cdot z = 0$$

Q2) $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

Prove that : $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \tan(u)$

Ans.

$$u = \sin^{-1}\left[\frac{x^2\left[1+(y/x)^2\right]}{x\left(1+y/x\right)}\right]$$

$$= \sin^{-1}\left[\frac{x\left(1+(y/x)^2\right)}{\left(1+y/x\right)}\right]$$

$$\Rightarrow \sin u = x\underbrace{\left[\frac{1+(y/x)^2}{1+y/x}\right]}_{\text{this becomes homogeneous}}$$

$$\Rightarrow \sin u = x\,\phi(y/x)$$

$\Rightarrow \sin u$ is a homogeneous function of degree 1.

By Euler's thm :-

~~$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 1(\sin$~~

$$\Rightarrow x\frac{\delta(\sin u)}{\delta x} + y\frac{\delta(\sin u)}{\delta y} = \sin u$$

$$\Rightarrow x\cos u\frac{\delta u}{\delta x} + y\cos u\frac{\delta u}{\delta y} = \sin u$$

$$\Rightarrow x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \tan u$$

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Q3) $z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Prove :- $x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = \sin 2z$

Ans. $z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan z = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\left(\frac{y}{x}\right)\right]}$$

$$\tan z = x^2 \phi(y/x)$$

** { ∴ By Euler's thm.,
tan z is a homogeneous function of degree 2, }

$$x\frac{\delta(\tan z)}{\delta x} + y\frac{\delta(\tan z)}{\delta y} = 2\tan z$$

$$\Rightarrow \sec^2 z\left(x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y}\right) = \frac{2\sin z}{\cos z}$$

$$\Rightarrow \frac{1}{\cos^2 z}\left(x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y}\right) = \frac{2\sin z}{\cos z}$$

$$\Rightarrow x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = 2\sin z\cos z = \sin 2z$$

Q4) $u = \log\left(\frac{x^4+y^4}{x+y}\right)$

Prove :- $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 3$

$$u = \log\left[\frac{x^4\left(1+\left(\frac{y}{x}\right)^4\right)}{x\left(1+\frac{y}{x}\right)}\right]$$

$$= \log\left[x^3\left(\frac{1+(y/x)^4}{1+y/x}\right)\right].$$

$$e^u = x^3\left(\frac{1+(y/x)^4}{1+y/x}\right) \qquad \left(\because e^{\log x} = x\right)$$

$$e^u = x^3\phi(y/x)$$

$\therefore$ $e^u$ is a function of $x^3$

$\therefore$ By Euler's thm

$$x\frac{\delta e^u}{\delta x} + y\frac{\delta e^u}{\delta y} = 3e^u$$

$$\Rightarrow e^u\left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right) = 3\cdot e^u$$

$$\Rightarrow x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 3$$

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Q5) $u = x^2 \log\left(\frac{y}{x}\right)$ verify Euler's thm.

Ans. $u = x^2 \log\left(\frac{y}{x}\right)$

$$\frac{\delta u}{\delta x} = x^2 \left(\frac{x}{y}\right)\left(\frac{-y}{x^2}\right) + \log\left(\frac{y}{x}\right) 2x$$

$$= -x + 2x \log\left(\frac{y}{x}\right)$$

$$\frac{\delta u}{\delta y} = x^2 \left(\frac{1}{y/x}\right)\left(\frac{1}{x}\right) = \frac{x^2}{y}$$

$$\therefore LHS = x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}$$

$$= x(-x) + 2x^2 \log\left(\frac{y}{x}\right) + x^2$$

$$= 2x^2 \log\left(\frac{y}{x}\right)$$

$$= 2u$$

Q6) $z = \sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)$. Verify Euler's thm.

Ans. $z = x^0 \left[\sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right] = x^0 \phi(y/x)$

$\therefore$ z is a homogeneous function of $(y/x)$ in degree 0.

$$\therefore x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = 0$$ (To prove)

$$\frac{\delta z}{\delta x} = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(\frac{-y}{x^2}\right) + \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= \frac{-y}{x\sqrt{x^2-y^2}} + \frac{y}{x^2+y^2}$$

$$\frac{\delta z}{\delta y} = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(\frac{1}{x}\right) + \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{1}{\sqrt{x^2-y^2}} - \frac{x}{x^2+y^2}$$

$$\text{LHS} = x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y}$$

$$= \frac{-y}{\sqrt{x^2-y^2}} + \frac{xy}{x^2+y^2} + \frac{y}{\sqrt{x^2-y^2}} - \frac{xy}{x^2+y^2}$$

$$= 0$$

$$= \text{RHS}$$

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Q. If $z = f(x, y)$ is a homogeneous function of degree $n$.

Then show that,

$$x^2 \frac{\delta^2 z}{\delta x^2} + 2xy \frac{\delta^2 z}{\delta x \delta y} + y^2 \frac{\delta^2 z}{\delta y^2} = n(n-1) z$$

Ans. Since $z = f(x, y)$ is a homogeneous function of ~~x & y~~ degree $n$.

By Euler's theorem,

$$x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = nz \quad \text{---} (1)$$

Diff. (1) w.r.t. $x$

$$x\frac{\delta^2 z}{\delta x^2} + \frac{\delta z}{\delta x} + y\frac{\delta^2 z}{\delta x \delta y} = n\frac{\delta z}{\delta x} \quad \text{---} (2)$$

Again diff. (1) w.r.t. $y$

$$x\frac{\delta^2 z}{\delta x \delta y} + y\frac{\delta^2 z}{\delta y^2} + \frac{\delta z}{\delta y} = n\frac{\delta z}{\delta y} \quad \text{---} (3)$$

(2) × $x$ + (3) × $y$

$$x^2\frac{\delta^2 z}{\delta x^2} + x\frac{\delta z}{\delta x} + xy\frac{\delta^2 z}{\delta x \delta y} + xy\frac{\delta^2 z}{\delta x \delta y} + y^2\frac{\delta^2 z}{\delta x^2} + y\frac{\delta z}{\delta y}$$

$$= nx\frac{\delta z}{\delta x} + ny\frac{\delta z}{\delta y}$$

$$\Rightarrow x^2\frac{\delta^2 z}{\delta x^2} + 2xy\frac{\delta^2 z}{\delta x \delta y} + y^2\frac{\delta^2 z}{\delta y^2} = x(n-1)\frac{\delta z}{\delta x} + y(n-1)\frac{\delta z}{\delta y}$$

$$= (n-1)\left[x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y}\right]$$

$$= (n-1)n z$$

$$= n(n-1)z$$

Q. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Prove:

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x\,\delta y} + y^2\frac{\delta^2 u}{\delta y^2} = 2(\cos 3u)(\sin u)$$

Ans. $\tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left(1+(y/x)^3\right)}{x\left[1-(y/x)\right]}$

$$= x^2\frac{\left(1+(y/x)^3\right)}{(1-y/x)}$$

⇒ $\tan u$ is a homogeneous function of degree 2

∴ By Euler's theorem:

$$x\frac{\delta(\tan u)}{\delta x} + y\frac{\delta(\tan u)}{\delta y} = 2\tan u$$

⇒ $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \sin 2u$ —— ①

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Diff. ① wrt $x$

$$x\frac{\delta^2 u}{\delta x^2} + \frac{\delta u}{\delta x} + y\frac{\delta u}{\delta x \delta y} = \cancel{\sin 2u}\ 2\cos 2u \frac{\delta u}{\delta x} \quad \text{——} ②$$

Diff. ① wrt $y$

$$x\frac{\delta u}{\delta x \delta y} + y\frac{\delta^2 u}{\delta y^2} + \frac{\delta u}{\delta y} = 2\cos 2u \frac{\delta u}{\delta y} \quad \text{——} ③$$

② $\times x$ + ③ $\times y$

$$x^2\frac{\delta^2 u}{\delta x^2} + x\frac{\delta u}{\delta x} + xy\frac{\delta u}{\delta x \delta y} + xy\frac{\delta u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} + y\frac{\delta u}{\delta y} = 2x\cos 2u \frac{\delta u}{\delta x} + 2y\cos 2u \frac{\delta u}{\delta y}$$

$$\Rightarrow x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = \cancel{\frac{\delta u}{\delta x} 2\cos 2u}$$

$$= x\frac{\delta u}{\delta x}(2\cos 2u - 1) + y\frac{\delta u}{\delta y}(2\cos 2u - 1)$$

$$= (2\cos 2u - 1)\left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right)$$

$$= (2\cos 2u - 1)(\sin 2u)$$

$$= 2\cancel{(\cos^2 u - \sin^2 u)}$$

$$= \left[2(\cos^2 u - \sin^2 u) - 1\right]\left[2\sin u \cos u\right]$$

$$= 4\sin u \cos^3 u - 4\sin^3 u \cos u - 2\sin u \cos u$$

$$= 2\sin u \cancel{\cos}\left[2\cos^3 u - 2\cos u \sin^2 u - 2\cos u\right]$$

$$= 2\cos 2u \sin 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$= 2\cos 3u \cdot \sin u$$

Q4) If $u = \log\left[\frac{x^2+y^2}{x+y}\right]$

find value of $x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x\delta y} + y^2\frac{\delta^2 u}{\delta y^2}$

Ans. $u = \log\left[\frac{x^2+y^2}{x+y}\right]$

$$e^u = \frac{x^2+y^2}{x+y} = \frac{x^2\left(1+\left(\frac{y}{x}\right)^2\right)}{x\left(1+\frac{y}{x}\right)} = x\frac{\left[1+\left(\frac{y}{x}\right)^2\right]}{\left(1+\frac{y}{x}\right)}$$

$\therefore$ ~~$\phi$~~ $\therefore e^u = x\phi(y/x)$

$\Rightarrow e^u$ is a homogeneous function of degree 1.

$\therefore$ By Euler's theorem

$$x\frac{\delta e^u}{\delta x} + y\frac{\delta e^u}{\delta y} = e^u$$

$$\Rightarrow x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 1 \quad \text{--- (1)}$$

Diff. (1) w.r.t $x$

$$x\frac{\delta^2 u}{\delta x^2} + \frac{\delta u}{\delta x} + y\frac{\delta u}{\delta x\delta y} = \cancel{\frac{\delta u}{\delta x}}\ 0 \quad \text{--- (2)}$$

Diff. (1) w.r.t $y$

$$x\frac{\delta u}{\delta x\delta y} + y\frac{\delta^2 u}{\delta y^2} + \frac{\delta u}{\delta y} = 0 \quad \text{--- (3)}$$

$② \times x + ③ \times y$

$$x^2\frac{\delta^2 u}{\delta x^2} + x\frac{\delta u}{\delta x} + xy\frac{\delta u}{\delta x \delta y} + xy\frac{\delta u}{\delta x\delta y} + y^2\frac{\delta^2 u}{\delta y^2} + y\frac{\delta u}{\delta y} = 0$$

$$\Rightarrow x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta u}{\delta x\delta y} + y^2\frac{\delta^2 u}{\delta y^2} = -\left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right)$$

$$= -1$$

Q5) $u = \operatorname{cosec}^{-1}\left\{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right\}^{1/2}$

Prove:- $x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x\delta y} + y^2\frac{\delta^2 u}{\delta y^2} = \frac{1}{144}\left(\tan^2 u + 13\right)$

Ans. $\operatorname{cosec} u = \left\{\frac{x^{1/2}\left(1+\left(\frac{y}{x}\right)^{1/2}\right)}{x^{1/3}\left(1+\left(\frac{y}{x}\right)^{1/3}\right)}\right\}^{1/2}$

$\frac{1}{2} - \frac{1}{3}$
$\overline{\quad 6 \quad}$

$$= x^{1/12}\left\{\frac{1+\left(\frac{y}{x}\right)^{1/2}}{1+\left(\frac{y}{x}\right)^{1/3}}\right\}^{1/2}$$

$\therefore \operatorname{cosec} u$ is a homogeneous function of degree $\frac{1}{12}$.

$$\therefore x\frac{\delta(\operatorname{cosec} u)}{\delta x} + y\frac{\delta(\operatorname{cosec} u)}{\delta y} = \frac{1}{12}\operatorname{cosec} u$$

$$\Rightarrow -\operatorname{cosec} u \cot u\left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right) = \frac{1}{12}\operatorname{cosec} u$$

$\Rightarrow$

$$\Rightarrow x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \frac{-1}{12}\tan u \quad \text{—①}$$

Diff. ① wrt $x$

$$x\frac{\delta^2 u}{\delta x^2} + \frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y\,\delta x} = \frac{-1}{12}\sec^2 u\left(\frac{\delta u}{\delta x}\right) \quad \text{—②}$$

Diff. ① wrt $y$

$$\Rightarrow x\frac{\delta u}{\delta x\,\delta y} + y\frac{\delta^2 u}{\delta y^2} + \frac{\delta u}{\delta y} = \frac{-1}{12}\sec^2 u\left(\frac{\delta u}{\delta y}\right) \quad \text{—③}$$

② $\times x$ + ③ $\times y$

$$\Rightarrow x^2\frac{\delta^2 u}{\delta x^2} + x\frac{\delta u}{\delta x} + xy\frac{\delta u}{\delta x\,\delta y} + xy\frac{\delta u}{\delta x\,\delta y} + y^2\frac{\delta^2 u}{\delta x^2} + y\frac{\delta u}{\delta y}$$

$$= \frac{-x}{12}\sec^2 u\frac{\delta u}{\delta x} - \frac{y}{12}\sec^2 u\frac{\delta u}{\delta y}$$

$$\Rightarrow x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta u}{\delta x\,\delta y} + y^2\frac{\delta^2 u}{\delta y^2}$$

$$= -\left[\frac{\delta u}{\delta x}x\left(\frac{1}{12}\sec^2 u + 1\right) + \frac{\delta u}{\delta y}y\left(\frac{1}{12}\sec^2 u + 1\right)\right]$$

$$= -\left(\frac{\sec^2 u + 1}{12}\right)\left\{x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right\}$$

$$= -\left(\frac{\sec^2 u + 12}{12}\right)\left\{\frac{-1}{12}\tan u\right\}$$

$$= \frac{1}{144}\left\{\sec^2 u \tan u + 12\tan u\right\}$$

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$$= \frac{1}{144} \tan u \left(\sec^2 u + 12\right)$$

$$= \frac{1}{144} \tan u \left(\tan^2 u + 1 + 12\right)$$

$$= \frac{1}{144} \tan u \left(\tan^2 u + 13\right)$$

Q6) $z = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y$

Show:- $x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = 6x^4y^2 \sin\left(\frac{x}{y}\right)$

Ans. $z = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right) + \log\left(\frac{x}{y}\right)$ —— (1)

$= u + v$

$u = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right)$

$= y^6\left[\left(\frac{x}{y}\right)^4 \sin^{-1}\left(\frac{x}{y}\right)\right]$

$\therefore$ $u$ is a homogeneous function of degree 6

$\therefore$ By Euler's thm,-

$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 6u$ —— (2)

$v = \log\left(\frac{x}{y}\right)$

$e^v = \frac{x}{y}$

$\therefore e^v$ is a function (homogeneous) of degree 0

By Euler's thm.

$x\frac{\delta e^v}{\delta x} + y\frac{\delta e^v}{\delta x} = 0 \cdot e^v$

$x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta x} = 0$ —— (3)

Adding (2) & (3)

$$x\left(\frac{\delta u}{\delta x}+\frac{\delta v}{\delta x}\right)+y\left(\frac{\delta u}{\delta y}+\frac{\delta v}{\delta y}\right) = 6u+0 = 6u$$

$$\Rightarrow x\frac{\delta z}{\delta x}+y\frac{\delta z}{\delta y} = 6x^4y^2\sin\left(\frac{x}{y}\right)$$

—x—x—x—x—x—

# LIMITS & CONTINUITY ...

## # Limit of a function of 2 variables

Let $f(x,y)$ be a function of 2 variable, then $f(x,y)$ is said to have a limit $L$ as the pt. $(x,y)$ approach towards $(a,b)$, if the value of $f(x,y)$ can be made as close as to the given finite no. $l$ for all the values of $(x,y)$ in approximately equal to $(a,b)$

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

Step 1) $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y)$

Step 2) $\lim\limits_{\substack{y\to b\\ x\to a}} f(x,y)$

If $(a,b) = (0,0)$

Step 3) Find limit along the path, $y = mx$

step 4) find limit along the path, $y = mx^{n}$ $n \to$ highest power of $x$ in the numerator

* In case of (0,0) all 4 steps are to be followed else only first 2 steps are to be followed

Q1) find $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)$

Ans

1. $\lim\limits_{\substack{x\to 0\\ y\to 0}} (x^2+y^2) = \lim\limits_{y\to 0} y^2 = 0$

2. $\lim\limits_{\substack{y\to 0\\ x\to 0}} (x^2+y^2) = \lim\limits_{x\to 0} x^2 = 0$

3. $\lim\limits_{\substack{x\to 0\\ y\to mx}} (x^2+m^2x^2) = \lim\limits_{x\to 0} x^2(1+m^2) = 0$

4. $\lim\limits_{\substack{x\to 0\\ y\to mx^2}} (x^2+m^2x^4) = \lim\limits_{x\to 0} x^2(1+m^2x^2) = 0$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0.$

Q2) $f(x,y) = \dfrac{y^2 - x^2}{x^2 + y^2}$ find limit at $(0,0)$

Ans 1. $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{y^2 - x^2}{x^2 + y^2} = \lim\limits_{y \to 0} \dfrac{y^2}{y^2} = 1$

2. $\lim\limits_{\substack{y \to 0 \\ x \to 0}} \dfrac{y^2 - x^2}{x^2 + y^2} = \lim\limits_{x \to 0} \dfrac{-x^2}{x^2} = -1$

$\because$ (1) $\neq$ (2)

$\therefore$ Limit does not exist at $(0,0)$

$\Rightarrow \lim\limits_{x \to 0} \left\{ \lim\limits_{y \to 0} f(x,y) \right\} \neq \lim\limits_{y \to 0} \left\{ \lim\limits_{x \to 0} f(x,y) \right\}$

Q3) $\lim\limits_{(x,y) \to (0,0)} \dfrac{x^2 y}{x^4 + y^2}$

Ans. 1. $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^2 y}{x^4 + y^2} = \lim\limits_{y \to 0} \dfrac{0}{y^2} = 0$

2. $\lim\limits_{\substack{y \to 0 \\ x \to 0}} \dfrac{x^2 y}{x^4 + y^2} = \lim\limits_{x \to 0} \dfrac{0}{x^4} = 0$

3. $$\lim_{\substack{x\to 0 \\ y\to mx}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2(mx)}{x^4+m^2x^2}$$

$$= \lim_{x\to 0} \frac{x^3(m)}{x^2(x^2+m^2)} = 0.$$

4. $$\lim_{\substack{x\to 0 \\ y\to mx^2}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2(mx^2)}{x^4+m^2x^4}$$

$$= \lim_{x\to 0} \frac{x^4(m)}{x^4(1+m^2)} = \frac{m}{1+m^2}$$

$\therefore$ Limit does not exist.

f. $$\lim_{(x,y)\to(0,0)} \frac{x^2+x-xy-y}{x-y}, \quad x\neq y$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x(x-y)+(x-y)}{x-y} = \lim_{(x,y)\to(0,0)} (x+1)$$

$= 1$

f. $\lim\limits_{(x,y)\to(3,-4)} x^2+xy+y^2$

$= 9 \cancel{4} -12+16$

$= 13$

g. $\lim\limits_{(x,y)\to(1,2)} \dfrac{2xy}{x^2+y^2}$

Ans. 1. $\lim\limits_{\substack{x\to1 \\ y\to2}} \dfrac{2xy}{x^2+y^2} = \lim\limits_{y\to2} \dfrac{2y}{1+y^2} = \dfrac{4}{5}$

2. $\lim\limits_{\substack{y\to2 \\ x\to1}} \dfrac{2xy}{x^2+y^2} = \lim\limits_{x\to1} \dfrac{4(x)}{x^2+4} = \dfrac{4}{5}$

$\therefore$ The limit of the function exists.

$\lim\limits_{(x,y)\to(1,2)} \dfrac{2xy}{x^2+y^2} = \dfrac{4}{5}$

# CONTINUOUS FUNCTIONS...

A function $f(x,y)$ is said to be continuous at $(a,b)$ if following conditions are true

(i) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists

(ii) $f(a,b)$ is defined

(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

If any 1 of the condition is not satisfied the function is not continuous

Q1) Discuss the continuity at origin

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Ans. 1. $\lim_{\substack{x\to 0 \\ y\to 0}} \dfrac{x^3-y^3}{x^2+y^2} = \lim_{y\to 0} \dfrac{-y^3}{y^2} = 0$

2. $\lim_{\substack{y\to 0 \\ x\to 0}} \dfrac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0} \dfrac{x^3}{x^2} = 0$

3. $$\lim_{\substack{x \to 0 \\ y \to mx}} \frac{x^3 - y^3}{x^2+y^2} = \lim_{x \to 0} \frac{x^3 - m^3 x^3}{x^2 + m^2 x^2}$$

$$= \lim_{x \to 0} x \frac{(1-m^3)}{(1+m^2)}$$

$$= 0$$

4. $$\lim_{\substack{x \to 0 \\ y \to mx^3}} \frac{x^3 - y^3}{x^2+y^2} = \lim_{x \to 0} \frac{x^3 - m^3 x^6}{x^2 + m^2 x^6}$$

$$= \lim_{x \to 0} x \frac{(1 - m^3 x^3)}{(1 + m^2 x^3)}$$

$$= 0$$

$\therefore \lim_{(x,y) \to (0,0)} f(x,y) = 0$

$\therefore$ The limit exists at $(0,0)$

$\therefore f(0,0) = 0$

$\therefore \lim_{(x,y) \to (0,0)} = f(0,0)$ is equal to $0$

$\therefore$ The function is continuous at $(0,0)$

Q. $f(x,y) = \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 2 & (x,y) = (0,0) \end{cases}$

Ans.

1. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x}{\sqrt{x^2+y^2}} = \lim\limits_{y\to 0} \dfrac{0}{y} = 0$

2. $\lim\limits_{\substack{y\to 0\\ x\to 0}} \dfrac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \dfrac{x}{x} = 1$

$\therefore (1) \neq (2)$

$\therefore$ The limit does not exist

$\therefore$ The function is not continuous

Q. $f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 3 & (x,y) = (0,0) \end{cases}$

Ans 1. $\lim\limits_{\substack{x\to 0 \\ y\to 0}} \dfrac{x^2}{\sqrt{x^2+y^2}} = \lim\limits_{y\to 0} \dfrac{0}{y} = 0$

2. $\lim\limits_{\substack{y\to 0 \\ x\to 0}} \dfrac{x^2}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \dfrac{x^2}{x} = 0$

3. $\lim\limits_{\substack{x\to 0 \\ y\to mx}} \dfrac{x^2}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \dfrac{x^2}{\sqrt{x^2+m^2x^2}}$

$= \lim\limits_{x\to 0} \dfrac{x^2}{x\sqrt{1+m^2}} = \lim\limits_{x\to 0} \dfrac{x}{\sqrt{1+m^2}} = 0$

4. $\lim\limits_{\substack{x\to 0 \\ y\to mx^2}} \dfrac{x^2}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \dfrac{x^2}{\sqrt{x^2+m^2x^4}}$

$= \lim\limits_{x\to 0} \dfrac{x^2}{\cancel{x}\sqrt{1+m^2x^2}}$

$= 0$

∴ The limit exists

$f(0,0) = 3$ is defined

$\lim_{(x,y)\to(0,0)} f(0,0) \neq 3$

$\therefore$ The function is not continuous

—— x —— x —— x —— x —— x —— x ——

## (Imp) MAXIMA & MINIMA of a function of two or more variables...

let $f(x,y)$ be a function of 2 variables.

Then if $f(a,b) > f(x,y)$ $\forall$ $(x,y) \in$ domain of $f$
$\Rightarrow$ $f$ has maximum value at $(a,b)$

or if $f(a,b) \leq f(x,y)$ $\forall$ $(x,y) \in$ domain of $f$.
$\Rightarrow$ then $f$ has minimum value of $(a,b)$,

## # SECOND ORDER PARTIAL DERIVATIVE TEST.

let $f(x,y)$ be a function which is to be maximised/minimised.

step i) Find $\frac{\delta f}{\delta x}$ & $\frac{\delta f}{\delta y}$

step 2) For initial points $\frac{\delta f}{\delta x} = 0$ & $\frac{\delta f}{\delta y} = 0$

let $(a,b)$ be a critical point

step 3) Find $r = \frac{\delta^2 f}{\delta x^2}$, $s = \frac{\delta^2 f}{\delta x \delta y}$, $t = \frac{\delta^2 f}{\delta y^2}$

step 4) Find $rt - s^2 \Big|_{(a,b)}$

I If $rt - s^2 \Big|_{(a,b)} > 0$ - the function has extremal value at $(a,b)$

a) If $r\Big|_{(a,b)} > 0$ - the function has minimum value at $(a,b)$

b) If $r\Big|_{(a,b)} < 0$ - the function has maximum value at $(a,b)$

II If $rt - s^2 \Big|_{(a,b)} < 0$ - the function has neither max. nor minimum

$(a,b)$ is known as saddle point

III If $rt - s^2 \Big|_{(a,b)} = 0$ then there is no conclusion

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Q1) $f(x,y) = x^3 + y^3 - 3xay$ , $a > 0$

Ans. (i) $\frac{\delta f}{\delta x} = 3x^2 - 3ay$

$\frac{\delta f}{\delta y} = 3y^2 - 3ax$

(ii) $\frac{\delta f}{\delta x} = 0$

$3x^2 - 3ay = 0$

$\Rightarrow y = \frac{x^2}{a}$

$\frac{\delta f}{\delta y} = 0$

$3y^2 - 3ax = 0$

$\Rightarrow x = \frac{y^2}{a}$

$\Rightarrow \left(\frac{x^2}{a}\right)^2 = ax$

$\Rightarrow x^4 = a^3 x$

$\Rightarrow x(x^3 - a^3) = 0$

$\Rightarrow x(x-a)(x^2 + a^2 + ax) = 0$

$\Rightarrow x = 0$ ; $x = a$

$y = 0$ ; $y = a$

$\therefore$ (0,0) & (a,a) are the two points

(iii) $r = \frac{\delta^2 f}{\delta x^2} = 6x$

$s = \frac{\delta^2 f}{\delta y \delta x} = -3a$

$t = \frac{\delta^2 f}{\delta y^2} = 6y$

$rt - s^2 = 36xy - 9a^2$

At $(0,0)$

$rt - s^2\big|_{(0,0)} = -9a^2 < 0$

$\Rightarrow$ function has neither max. nor min. at $(0,0)$

At $(a,a)$

$rt - s^2 = 36xy - 9a^2$

$rt - s^2\big|_{(0,0)} = 36a^2 - 9a^2 = 27a^2 > 0$

$\Rightarrow$ function has extremal value at $(0,0)$

Now $r\big|_{(a,a)} = 6a > 0 \quad (\because a > 0)$

$\Rightarrow$ function has minimum value at $(a,a)$.

Minimum value $= f(a,a)$

$= a^3 + a^3 - 3a^3$

$= -a^3$

— x — x — x — x —

$r\big|_{(a,a)} = 6a > 0 \quad \text{if } a > 0$

$< 0 \quad \text{if } a < 0$

$\therefore$ function has maximum value at $(-a,-a)$

$\therefore f(-a,-a) = -a^3 - a^3 - 3a^3$

$= -5a^3$

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Q2) $f(x,y) = 3x^2 - 2xy + y^2 - 8y$

Ans. (i) $\frac{\delta f}{\delta x} = 6x - 2y$ $\frac{\delta f}{\delta y} = -2x + 2y - 8$

(ii) $\frac{\delta f}{\delta x} = 0$

$6x = 2y$

$\Rightarrow y = 3x$

$\frac{\delta f}{\delta y} = 0$

$y - x = \cancel{8}\ 4$

$y = \cancel{8 + x}\ 4 + x$

$\therefore 3x = \cancel{8}\ 4 + x$

$2x = \cancel{8}\ 4$

$x = 2$

$y = 6$

$\therefore (2,6)$ is a critical point

(iii) $r = \frac{\delta^2 f}{\delta x^2} = 6$ $s = \frac{\delta^2 f}{\delta x \delta y} = -2$

$t = 2$

(iv) $rt - s^2 \Big| = 12 - 4 = 8$

$\Rightarrow rt - s^2 \Big|_{(2,6)} = 8 > 0$

$\Rightarrow$ f has extremal value at $(2,6)$

$r\Big|_{(2,6)} = 6 > 0$

aminotes

$\Rightarrow f$ has minimum value at $(2,6)$

$$f(2,6) = 12 - 24 + 36 - 48$$
$$= -24$$

Q3) $f(x,y) = 4xy - x^4 - y^4$

Ans. (i) $\frac{\delta f}{\delta x} = 4y - 4x^3$ $\qquad$ $\frac{\delta f}{\delta y} = 4x - 4y^3$

(ii) $\frac{\delta f}{\delta x} = 0$ $\qquad$ $\frac{\delta f}{\delta y} = 0$

$\Rightarrow 4y = 4x^3$

$\Rightarrow y = x^3$ $\qquad$ $\Rightarrow x = y^3$

$$x = (x^3)^3$$
$$\Rightarrow x^9 - x = 0$$
$$\Rightarrow x(x^8 - 1) = 0$$
$$\Rightarrow x(x^4-1)(x^4+1) = 0$$
$$\Rightarrow x(x^2-1)(x^2+1)(x^4+1) = 0$$
$$\Rightarrow x(x-1)(x+1)(x^2+1)(x^4+1) = 0$$

$x = 0 \; ; y = 0$ $\qquad$ or

$x = 1 \; ; y = 1$

$x = -1 \quad y = -1$

(iii) $r = \frac{\delta^2 f}{\delta x^2} = -12x^2$ $\quad s = \frac{\delta^2 f}{\delta x \delta y} = 4$

$t = \frac{\delta^2 f}{\delta y^2} = -12y^2$

(iv) $rt - s^2 = 144x^2y^2 - 16$

for (0,0)

$rt - s^2 \big|_{(0,0)} = -16 < 0$

$\Rightarrow$ function has neither max. nor minima

for (1,1)

$rt - s^2 \big|_{(1,1)} = 144 - 16 = 128 > 0$

$\Rightarrow$ function has extremal value at (1,1)

$r\big|_{(1,1)} = -12 < 0$

$\Rightarrow$ f has maximum value at (1,1)

$f(1,1) = 4 - 1 - 1 = 2.$

for (-1,-1)

$rt - s^2 \big|_{(-1,-1)} = 128 > 0$

$r\big|_{(-1,-1)} = -12 < 0$

$\Rightarrow$ f has maximum value at (-1,-1)

$f(-1,-1) = 4 - 1 - 1 = 2$

Q4) $f(x,y) = 8x^3 - 24xy + y^3$

Ans (i) $\frac{\delta f}{\delta x} = 24x^2 - 24y$ $\qquad \frac{\delta f}{\delta y} = 3y^2 - 24x$

(ii) $\frac{\delta f}{\delta x} = 0$ $\qquad \frac{\delta f}{\delta y} = 3y^2 - 24x = 0$

$24x^2 = 24y$ $\qquad \Rightarrow y^2 = 8x$

$x^2 = y$

$\therefore 8x = x^4$

$\Rightarrow x^4 - 8x = 0$

$\Rightarrow x(x^3 - 8) = 0$

$\Rightarrow x(x-2)(x^2 + 4 + 4x) = 0$

$\therefore x = 0, \; y = 0$

$x = 2, \; y = 4$

$\therefore (0,0)$ & $(2,4)$ are the 2 critical points.

(iii) $r = \frac{\delta^2 f}{\delta x^2} = 48x$

$s = \frac{\delta^2 f}{\delta x \delta y} = -24$

$t = \frac{\delta^2 f}{\delta y^2} = 6y$

$rt - s^2 = 288xy - (-24)^2$
$= 288xy - 576$

for (0,0)
$rt - s^2 = -576 < 0$
$\Rightarrow$ At (0,0), it has neither max. nor minimum

For (2,4)
$rt - s^2 = 288(8) - 576$
$= 1728 > 0$
$\Rightarrow$ At (2,4) the function has extremal values

$r|_{(2,4)} = 48(2) = 96 > 0$

$\Rightarrow$ The function has minimum value

$f(2,4) = 8(8) - 24(8) + 64$
$= -64$ is the minimum value

# APPLICATIONS of MAXIMA & MINIMA...

(**) Imp

Q. Find the dimensions of a rectangular box open at the top, having vol. of 32 cubic units & requiring least material for its construction.

Ans. let $x, y, z$ be dimensions of box

Since volume is 32

$\therefore xyz = 32$ —— ①

$\Rightarrow \left( z = \frac{32}{xy} \right)$

Surface area $= xy + 2yz + 2zx$ —— ②

Putting value of ① in ②

$S.A. = xy + 2y\left(\frac{32}{xy}\right) + 2x\left(\frac{32}{xy}\right)$

$= xy + \frac{64}{x} + \frac{64}{y}$

let $f(x,y) = xy + \frac{64}{x} + \frac{64}{y}$

(i) $\frac{\delta f}{\delta x} = y - \frac{64}{x^2}$ $\qquad$ $\frac{\delta f}{\delta y} = x - \frac{64}{y^2}$

(ii)

| | |
|---|---|
| $\frac{\delta f}{\delta x} = 0$ | $\frac{\delta f}{\delta y} = 0$ |
| $y = \frac{64}{x^2}$ | $x = \frac{64}{y^2}$ |
| | $\Rightarrow y^2 = \frac{64}{x}$ |

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$$\Rightarrow \frac{64}{x} = \frac{(64)^2}{x^4}$$

$$\Rightarrow x^4 = 64x$$

$$\Rightarrow x(x^3 - 64) = 0$$

$$\Rightarrow x(x-4)(x^2 + 16 + 8x) = 0$$

$x = 0$ , $y = \infty$ (Not possible)

$x = 4$ , $y = 4$

$\therefore (4, 4)$ is a critical point.

(iii) $r = \frac{\partial^2 f}{\partial x^2} = -64\left(\frac{-2}{x^3}\right) = \frac{128}{x^3}$

$s = \frac{\partial^2 f}{\partial x \partial y} = 1$ ; $t = \frac{\partial^2 f}{\partial y^2} = \frac{128}{y^3}$

$$rt - s^2 = \frac{128 \times 128}{x^3 y^3} - 1$$

$$rt - s^2 \Big|_{(4,4)} = \frac{\cancel{128}^{\,2} \times \cancel{128}^{\,2}}{\cancel{64} \times \cancel{64}} - 1 = 4 - 1 = 3 > 0$$

$\Rightarrow$ It has an extremal value at $(4,4)$

$$r\Big|_{(4,4)} = \frac{128}{64} = 2 > 0$$

$\Rightarrow$ The function has minimum value

$\therefore$ for least material,

$$x = 4, \; y = 4, \; z = \frac{32}{16} = 2$$

# # LEGRANGE'S METHOD of MULTIPLIERS...

Adv: no need to check whether function is maximal minima

Let $f(x,y)$ be a function which is to be maximis-ed / minimis-ed and let $g(x,y) = 0$ be a given constraint

let $F(x,y) = f + \lambda g$, when $\lambda \in Z$

$\frac{\delta F}{\delta x} = 0$ ; $\frac{\delta F}{\delta y} = 0$ ; $\frac{\delta F}{\delta z} = 0$

Q1) Find dimensions of a box having surface area 108, having max. volume.

Ans. Let the sides be $x, y, z$

$108 = xy + 2yz + 2zx$

let $f(x,y,z) = xyz$ (function to be max/minimised)

let $g(x,y,z) = xy + 2zy + 2zx = 108$ ⟵ (17)

let $F(x,y,z) = f + \lambda g$ ($\lambda$ can be determined)

$F = xyz + \lambda(xy + 2yz + 2zx)$ ⟵ (18)

$$\frac{\delta F}{\delta x} = yz + \lambda(y+2z) = 0 \quad \text{——} ① \Rightarrow \lambda = \frac{-yz}{y+2z}$$

$$\frac{\delta F}{\delta y} = xz + \lambda(x+2z) = 0 \quad \text{——} ② \Rightarrow \lambda = \frac{-xz}{x+2z}$$

$$\frac{\delta F}{\delta z} = xy + 2\lambda(y+x) = 0 \quad \text{——} ③ \Rightarrow \lambda = \frac{-xy}{2(x+y)}$$

~~From eqn. ① & ②~~

$$\cancel{yz + \lambda(y+2z) = xz + \lambda(x+2z)}$$

$$\cancel{\lambda(y-x) = z(x-y)}$$

$$\cancel{\therefore \lambda = -z}$$

From ① & ②

$$\frac{\cancel{-}y\cancel{z}}{y+2z} = \frac{\cancel{-}x\cancel{z}}{x+2z}$$

$$\cancel{xy} + 2yz = \cancel{xy} + 2xz$$

$$\Rightarrow x = y$$

From ② & ③

$$\frac{\cancel{-x}z}{x+2z} = \frac{\cancel{-x}y}{2(x+y)}$$

$$2zx + \cancel{2zy} = xy + \cancel{2zy}$$

$$\Rightarrow y = 2z$$

From ① & ③

$$\frac{\cancel{-y}z}{y+2z} = \frac{\cancel{-}x\cancel{y}}{2(x+y)} \Rightarrow \cancel{2zx} + 2yz = xy + \cancel{2zx}$$

$$\Rightarrow y$$

$\therefore x = y = 2z$

Putting in (A)

$$108 = x^2 + 2x\left(\frac{x}{2}\right) + 2(x)\left(\frac{x}{2}\right)$$

$$108 = 3x^2$$

$\Rightarrow x^2 = 36$

$\Rightarrow x = \pm 6$

For $x = 6, y = 6, z = 3$

$x = -6, y = -6, z = -3$ (Rejected, $\because$ dimensions can't be negative)

$\therefore$ Dimensions of the box are 6, 6, 3

Q2) The sum of 3 no's is constant, prove that their product is max. when they are equal.

let the 3 no's be $x, y, z$

let $f(x,y,z) = xyz$

Sum $= x + y + z = k$

let $g(x,y,z) = x + y + z - k$

$F(x,y,z) = f(x,y,z) + \lambda \phi g.(x,y,z)$ (value of $\lambda$ can be determined)

$F(x,y,z) = xyz + \lambda(x+y+z-k)$

$\frac{\delta f}{\delta x} = yz + \lambda = 0 \qquad \lambda = -yz$ —— ①

$\frac{\delta f}{\delta y} = xz + \lambda = 0 \qquad \lambda = -xz$ —— ②

$\frac{\delta f}{\delta z} = xy + \lambda = 0 \qquad \lambda = -xy$ —— ③

from ① & ② $\qquad$ from ② & ③

$-yz = -xz \qquad -xz = -xy$

$x = y \qquad y = z$

$\therefore x = y = z$

∴ Their product is max, when they are equal

Q3) Find the pts. on the plane $x+y+2z=5$, that is close to the pt. $(0,3,4)$

dist is min.

Ans.

let $Q(x,y,z)$ be a point on the plane

$PQ = \sqrt{x^2+(y-3)^2+(z-4)^2}$

Let $f(x,y,z) = x^2+(y-3)^2+(z-4)^2$ —— ①

Let $g(x,y,z) = x+y+2z-5$ —— ②

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$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z) \quad \text{—— (3)}$$

$$= x^2+(y-3)^2+(z-4)^2 + \lambda(x+y+2z-5)$$

$$\frac{\delta F}{\delta x} = 2x+\lambda \Rightarrow \lambda = -2x \quad \text{—— (1)}$$

$$\frac{\delta F}{\delta y} = 2(y-3)+\lambda \Rightarrow \lambda = 6-2y \quad \text{—— (2)}$$

$$\frac{\delta F}{\delta z} = 2(z-4)+2\lambda \Rightarrow \lambda = 4-z \quad \text{—— (3)}$$

from (1), (2) & (3)

$-2x = 6-2y$ | $2x = z-4$

$x = y-3$ | $z = 2x+4$

$y = x+3$

$$y = x+3 \; ; \; z = 2x+4$$

Putting these values in (2)

$$x+y+2z = 5$$

$$x+x+3+4x+8 = 5$$

$$6x = -6$$

$$x = -1$$

$$x = -1, \; y = 2, \; z = 2$$

$\therefore$ The points are $(-1,2,2)$

Q4) Divide 24 into 3 parts such that product of first, sq. of second & cube of third is maximum.

Ans. let the 3 numbers be $x, y, z$

let $g(x,y,z) = x+y+z-24$ —— (A)

let $f(x,y,z) = xy^2z^3$ (function to be made maximum)

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$
$$= xy^2z^3 + \lambda(x+y+z-24)$$

$$\frac{\delta F}{\delta x} = y^2z^3 + \lambda = 0 \qquad \lambda = -y^2z^3 \quad \text{—— (1)}$$

$$\frac{\delta F}{\delta y} = 2xyz^3 + \lambda = 0 \qquad \lambda = -2xyz^3 \quad \text{—— (2)}$$

$$\frac{\delta F}{\delta z} = 3xy^2z^2 + \lambda = 0 \qquad \lambda = -3xy^2z^2 \quad \text{—— (3)}$$

From (1), (2) & (3)

$$y^{\cancel{2}}\cancel{z^3} = 2x\cancel{y}\cancel{z^3} \qquad \cancel{y^2}z^{\cancel{3}} = 3x\cancel{y^2}\cancel{z^2}$$
$$y = 2x \qquad z = 3x$$

$\therefore \quad y = 2x, z = 3x$

Putting these values in (A)

$$x + 2x + 3x = 24$$
$$6x = 24$$
$$\Rightarrow x = 4, y = 8, z = 12$$

$\therefore$ The reqd. no's are $4, 8, 12$

Q5) Divide 120 into 3 parts, such that sum of their products 2 at a time is maximum.

Ans: $g(x,y,z) = x+y+z-120$ —— (A)

$f(x,y,z) = xy+yz+zx$ (function to be maximised)

$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

$= xy+yz+zx+\lambda(x+y+z-120)$

$\frac{\delta F}{\delta x} = y+z+\lambda \Rightarrow \lambda = -(y+z)$ —— (1)

$\frac{\delta F}{\delta y} = x+z+\lambda \Rightarrow \lambda = -(x+z)$ —— (2)

$\frac{\delta F}{\delta z} = x+y+\lambda \Rightarrow \lambda = -(x+y)$ —— (3)

from (1), (2) & (3)

$y+z = x+z$ ; $x+z = x+y$

$\Rightarrow x = y$ ; $\Rightarrow y = z$

$\therefore x = y = z$

Putting this in (A)

$120 = 3x$

$x = 40$

$\therefore x = y = z = 40$

$\therefore$ The 3 no's are 40, 40, 40

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(R) Q6) ** The temp. T at any pt $(x,y,z)$ in the space is $T = 400xyz^2$. find the max. temp. on the surface of unit sphere.

Ans. $g(x,y,z) = x^2+y^2+z^2-1$ —— (A)

$f(x,y,z) = 400xyz^2$ (quantity to be maximised)

$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

$= 400xyz^2 + \lambda(x^2+y^2+z^2-1)$

$\frac{\delta F}{\delta x} = 400yz^2 + 2x(\lambda) \Rightarrow \lambda = -\frac{400yz^2}{2x}$ —— (1)

$\frac{\delta F}{\delta y} = 400xz^2 + \lambda(2y) \Rightarrow \lambda = \frac{-400xz^2}{2y}$ —— (2)

$\frac{\delta F}{\delta y} = 800xyz + \lambda(2z) \Rightarrow \lambda = -\frac{800xyz}{2z}$ —— (3)

From (1), (2) & (3)

$\frac{\cancel{400}y\cancel{z^2}}{2x} = \frac{\cancel{400}xz^2}{\cancel{2}y}$

$y^2 = x^2$

$y = \pm x$

~~$x = \pm y$~~

$\frac{400\cancel{y}z^2}{2x} = \frac{800x\cancel{y}\cancel{z}}{\cancel{2}z}$ (2)

$z^2 = 2x^2$

$z = \pm\sqrt{2}x$

$(x, x, \sqrt{2}x)$, $(x, -x, -\sqrt{2}x)$ are the 2 points

Putting in (A)

$x^2 + x^2 + 2x^2 = 1$

$5x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{5}}$

$\therefore x = \pm \frac{1}{55}$, $y = \pm \frac{1}{55}$, $z = \pm\sqrt{\frac{2}{5}}$

# Vector Calculus:

**Scalar Field** - If a scalar quantity $\phi$ is a function of a pt. $(x,y,z)$ that depends on its position in space is called a Scalar pt function. It does not depend upon the choice of the co-ordinate system. Ex, - temp, density etc.

**Vector Pt func.** - A function $\vec{V}$ which depends on its value and its position in space is called a vector pt. function. Ex - Acc.

**Del Operator: (nabla)** The Del operator denoted by $\nabla$ is $= \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

If the Del operator is applied on a vector pt. function, the result as scalar and if applied to a scalar pt. function results in a vector itself.

**Gradient** Let $\phi(x,y,z)$ be a scalar pt. function, then the gradient of this function is denoted as $\text{grad}\,\phi = \nabla\phi$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x,y,z)$$

(Also a ~~unit~~ vector to the surface $\phi$).

$$\Rightarrow \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

Ex Find the grad $\phi(x,y,z)$ at $(1,-2,-1)$ ; $\phi(x,y,z) = x^3+y^3+3xyz$.

$$\text{grad}\,\phi(x,y,z) = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\hat{i}(3x^2+3yz) + \hat{j}(3y^2+3xz) + \hat{k}(3xy)$$

$12+$ at $(1,-2,-1)$, it will be

$3+$ $$\hat{i}(9) + \hat{j}(9) + \hat{k}(-6)$$

$$9\hat{i} + 9\hat{j} - 6\hat{k}$$

Ex $\phi = r^m$. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ grad $\phi$ = ?

$$r = (x^2+y^2+z^2)^{1/2}$$

$$r^m = (x^2+y^2+z^2)^{m/2} = \phi$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

$$\hat{i}\left(\frac{m x}{\cancel{2}}(x^2+y^2+z^2)^{\frac{m}{2}-1}\right) + \hat{j}\left(my\,(x^2+y^2+z^2)^{m/2-1}\right) + \hat{k}\left(mz\,(x^2+y^2+z^2)^{\frac{m}{2}-1}\right)$$

$$m(x^2+y^2+z^2)^{\frac{m}{2}-1}(\vec{r})$$ $\boxed{m \times r^{m-2}\,\vec{r}}$

$\frac{\vec{r}}{|r|} = \hat{r}$

$\vec{r} = \hat{r}\,r$

$\cancel{\frac{\partial(r^m)}{\partial r} \times r} \Rightarrow \cancel{mr^{m-1}}$

$$\boxed{mr^{m-1}\hat{r}}$$

Geometrical significance of gradient func: $\nabla\phi$ is a vector normal to the surface $\phi$

Eg unit vector $\perp$ to $x^3+y^3+3xyz=\phi$ at $(1,2,-1)$

$\nabla\phi$ is a vector $\perp$ to $\phi$.

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

$$\nabla\phi = \hat{i}(3x^2+3xyz) + \hat{j}(3y^2+3xz) + \hat{k}(3xy)$$

$$\nabla\phi = \hat{i}(-3) + \hat{j}(9) + 6\hat{k}$$

$$-3\hat{i}+9\hat{j}+6\hat{k} \qquad |\nabla\phi| = \sqrt{9+81+36}.$$

$$\text{unit vector} = \frac{\nabla\phi}{|\nabla\phi|} \Rightarrow \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{\sqrt{9+81+36}}.$$

Eg If $\phi$ is an acute angle b/w $x^2yz = 3x+z^2$ and $3x^2-y^2+2z=1$ at $(1,-2,1)$.

$\phi_1$: $x^2yz-3x-z^2=0$ $\qquad$ $3x^2-y^2+2z-1=0$.

$\perp$ to $\phi_1 \Rightarrow \nabla\phi_1$

$$\nabla\phi_1 = \hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}.$$

$$\hat{i}(2xyz-3) + \hat{j}(x^2z) + \hat{k}(x^2y-2z)$$

at $(1,-2,1)$. $\qquad -2-2$

$$= \hat{i}(-7) + \hat{j} + (-4)\hat{k}. \qquad \nabla\phi_1\cdot\nabla\phi_2 = -42+4-8$$

$$\nabla\phi_2 = \hat{i}(6x) + \hat{j}(-2y) + \hat{k}(2)$$

at $(1,-2,1) \Rightarrow 6\hat{i}+4\hat{j}+2\hat{k}$.

Now angle b/w $\phi_1$ and $\phi_2$ is equal to the angle b/w.

$\nabla\phi_1$ and $\nabla\phi_2$ as they are vectors $\perp$ to these

scalar pt. functions

Angle b/w two vectors $\Rightarrow$ $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$\boxed{\Rightarrow \frac{-46}{(\sqrt{4+9+16+1})(\sqrt{36+16+4})}}$$

Ex Calculate the angle between $\perp$ vectors on $xy = z^2$ at $(4,1,2)$ & $(3,3,-3)$

$xy - z^2 = 0.$

$\perp$ vector on $\phi \Rightarrow$ grad $\phi$.

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

$\hat{i}(y) + \hat{j}(x) + \hat{k}(-2z)$

at $(4,1,2)$

$\hat{i} + 4\hat{j} - 4\hat{k}$

at $(3,3,-3)$

$3\hat{i} + 3\hat{j} + 6\hat{k}$

$\nabla\phi_1 \cdot \nabla\phi_2 = 3 + 12 - 24$

$\Rightarrow -9.$

36
18
3|54
3|18
6

$\sqrt{33} \times \sqrt{54}$

$\sqrt{33} \times 3\sqrt{6}$

angle b/w two vectors $\Rightarrow$ $\cos\theta = \dfrac{\nabla\phi_{(1)} \cdot \nabla\phi_{(2)}}{|\nabla\phi_{(1)}||\nabla\phi_{(2)}|}$ $\Rightarrow \dfrac{-\cancel{9}^{3}}{\cancel{3}\sqrt{6\times3\times11}}$

$\dfrac{-1}{\sqrt{22}}$

Directional Deriviative of a scalar pt function $\phi(x,y,z)$ at pt. P. in the direction of unit vector $\hat{\alpha}$ is given by

$\nabla\phi\cdot\hat{\alpha}$

Ex Find the directional deriviative of $\phi = 2xy + z^2$ at $P(1,-1,3)$ in the direction of $\vec{\alpha} = \hat{i} + 2\hat{j} + 2\hat{k}$.

$\nabla\phi$ is the vector $\perp$ to $\phi$.

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

$$= \hat{i}(2y) + \hat{j}(2x) + \hat{k}(2z)$$

$\Rightarrow$ at $(1,-1,3) \Rightarrow -2\hat{i} + 2\hat{j} + 6\hat{k}$.

in direction of $\hat{\alpha} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$

$$\nabla\phi\cdot\hat{\alpha} \Rightarrow \frac{-2+4+12}{3} \Rightarrow \frac{14}{3}.$$

Ex Find directional deriviative of $\phi_1 = xy^2 + yz^3$ at $(1,-1,1)$ along the direction normal to the surface $x^2 + y^2 + z^2 - 9 = 0$ at $(1,2,2)$

$\phi_2$

$\vec{\alpha} = \nabla\phi_2$ at $(1,2,2)$

$$\nabla\phi_1 = \hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}.$$

$\Rightarrow \nabla\phi_1 \cdot \dfrac{\nabla\phi_2}{|\nabla\phi_2|}$

$$\hat{i}(y^2) + \hat{j}(2xy + z^3) + \hat{k}(3yz^2)$$

$(1,-1,1)$

$$\hat{i} + (-1)\hat{j} + (-3)\hat{k} \rightarrow \nabla\phi_1$$

$$\nabla\phi_2 = \hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}$$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$

at $(1,2,2)$

$$2\hat{i} + 4\hat{j} + 4\hat{k} = \vec{x}$$

$$\Rightarrow \nabla\phi_1 \cdot \hat{x} \rightarrow \frac{(\hat{i} - \hat{j} - 3\hat{k})(2\hat{i} + 4\hat{j} + 4\hat{k})}{\sqrt{36}}$$

$$\frac{2-4-12}{6} \Rightarrow \frac{-14}{6} = -7/3.$$

## Divergence of a vector point function:

If $\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is a vector point function, then its divergence is given as $\nabla\cdot\vec{V} = \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)$

If my divergence of $\vec{V} \Rightarrow \nabla\cdot\vec{V} = 0$, then the function is solenoidal.

## Physical Interpretation of Divergence:

It gives us the rate of outflow per unit volume at a point in of the fluid.

$\rightarrow$

Ex. Find the divergence of $\vec{F}(x,y,z) = e^{xyz}(x^2y\hat{i} + yz^2\hat{j} + zx^2\hat{k})$ at $(2,2,2)$

$$\nabla\cdot\vec{V} = \left(\hat{i}\frac{\partial V_x}{\partial x} + \hat{j}\frac{\partial V_y}{\partial y} + \hat{k}\frac{\partial V_z}{\partial z}\right)$$

$$\hat{i}\left(yze^{xyz}(x^2y) + 2e^{xyz}xy\right) + \hat{j}\left(xze^{xyz}(yz^2) + z^2e^{xyz}\right) +$$

$$\hat{k}\left(xye^{xyz}(zx^2) + x^2e^{xyz}\right)$$

at $(2,2,2)$

$$e^8\left(\hat{i}(32+8) + \hat{j}(32+4) + \hat{k}(32+4)\right)$$

$$e^8(40\hat{i} + 36\hat{j} + 36\hat{k}) \qquad (102)e^8$$

Ex. $F = (-x^2+yz)\hat{i} + (4y - z^2x)\hat{j} + (12xz - 4z)\hat{k}$ solenoidal or not?

$$\nabla\cdot\vec{F} \Rightarrow \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$(-2x) + (4) + (12x - 4)$$

$$\nabla\cdot\vec{F} = -2x + 4 + 12x - 4 \Rightarrow 10x. \text{ Not solenoidal}$$

## Curl of a Vector pt. function:

If $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is a vector point function, then cross product of it with $\nabla$ ~~function~~ operator is nothing but the curl of a function.

$$\text{Curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

i) If the Curl $\vec{V} = 0$, then the vector is irrotational. (There is no rotation in the vector.)

Ex $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$. Find Curl $\vec{V}$ at $(2,-1,1)$

$$\text{Curl}\vec{V} = \nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$\hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$\hat{i}(-2yz - 0) - \hat{j}(z^2 - xy) + \hat{k}(6xy - xz)$$

at $(2,-1,+1)$

$$2\hat{i} - \hat{j}(8) + \hat{k}(-14) = 2\hat{i} - \cancel{8}3\hat{j} - 14\hat{k}.$$

Q. $\vec{V} = (z+y)\hat{i} + \hat{j}(x+z) + \hat{k}(x+y)$ Curl $\vec{V}$ is irrotational or not?

$$\text{Curl}\,\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$\Rightarrow \hat{i}(1-1) - j(1-1) + \hat{k}(1-1)$$

$$\Rightarrow 0.$$

$\therefore$ Curl $\vec{V} = 0 \Rightarrow$ the plane is irrotational.

Q. $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz) \to \phi$.

Find div $\vec{F}$ and Curl $\vec{F}$.

$$\vec{F} = \nabla\phi \Rightarrow \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

$$\hat{i}(3x^2 - 3yz) + \hat{j}(3y^2 - 3xz) + \hat{k}(3z^2 - 3xy).$$

$$\text{div}\,\vec{F} \Rightarrow \nabla\cdot\vec{F} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}.$$

$$6x + 6y + 6z \Rightarrow 6(x+y+z)$$

$$\begin{bmatrix} 0 & -4 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x_2 + x_3 = 0$

$-2x_2 = 0$

$x_3 = 0$

$x_1 = l$

$x_3 = 0$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} l \quad \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \hat{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right).$$

$$= \hat{i}(-3x + 3x) - \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z$$

$$\Rightarrow 0.$$

$\text{Curl } \vec{F} = 0 \implies$ The plane is irrotational

$\vec{V} = x^2 y\,\hat{i} + y^2 z\,\hat{j} + z^2 y\,\hat{k}$ $\quad \text{Curl}(\text{curl } \vec{V}) = ?$

$$\text{Curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}. \\ V_x & V_y & V_z \end{vmatrix}$$

$$\hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) - \hat{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)$$

$$\hat{i}(z^2 - y^2) + \hat{k}(0 - x^2) - \hat{j}(0 - 0)$$

$$\vec{V'} \Rightarrow (z^2 - y^2)\hat{i} - x^2\hat{k}$$

$$\text{Curl } \vec{V'} = \vec{\nabla} \times \vec{V'} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right).$$

$= \hat{i}(0-0) - \hat{j}(-2x-2z) + \hat{k}(-2y)$

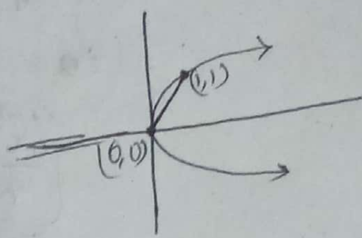
$2\left((x+z)\hat{j} + y\hat{k}\right)$

Line Integral:

Any integral which can be evaluated along a curve is said to be line integral.

$\boxed{\text{Conservative Field} \Rightarrow W = 0}$

Eg Find $\int_C \vec{F} \cdot dr$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and c is $y^2 = x$ joining (0,0) to (1,1)




$r = x\hat{i} + y\hat{j}$

$dr = dx\hat{i} + dy\hat{j}$

$$\int_{(0,0)}^{(1,1)} \vec{F} \cdot dr$$

$$\int_{(0,0)}^{(1,1)} (x^2\hat{i} + xy\hat{j})(dx\hat{i} + dy\hat{j})$$

$$\int_{(0,0)}^{(1,1)} x^2 dx + xy\,dy$$

$\because y = \sqrt{x}$ and $2y\,dy = dx \Rightarrow dy = \frac{dx}{2y}$

$$\int_0^1 x^2 dx + x\sqrt{x}\frac{dx}{2\sqrt{x}}$$

$$\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{4}\right]_0^1 \Rightarrow \frac{1}{3} + \frac{1}{4} = \boxed{\frac{7}{12}}$$

Ex. Calculate $\int_C \vec{F}\cdot d\vec{r}$ ⇒ $F = 3xy\,\hat{i} - y^2\hat{j}$ along C where C is

$y = 2x^2$ from (0,0) to (1,2)

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\int_C (3xy\,\hat{i} - y^2\hat{j})(dx\,\hat{i} + dy\,\hat{j})$$

$$\int_{(0,0)}^{(1,2)} 3xy\,dx - y^2 dy$$

$\because y = 2x^2$ and $dy = 4x\,dx$.

$$\int_{(0)}^{(1)} 6x^3 dx - 4x^4 \times 4x\,dx$$

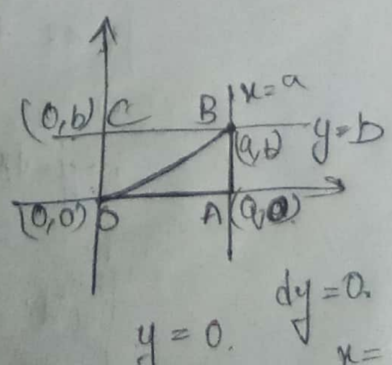
$$\left[\frac{6x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_0^1$$

$$\frac{3\cancel{6}}{_2\cancel{4}} - \frac{\cancel{16}\,8}{\cancel{6}_3} \Rightarrow \frac{3}{2} - \frac{8}{3} \Rightarrow \boxed{\frac{-7}{6}}$$

Ex. $(x^2+y^2)\hat{i} - 2xy\hat{j} = \vec{F}$. Calculate work done in moving a point from (0,0) to (a,b) along the rectangle bounded by the lines.

$x = 0,\ x = a,\ y = 0,\ y = 1$

$\int \vec{F}\cdot d\vec{r}$ $\quad d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$




$$\int x^2 dx + y^2 dx - 2xy\,dy$$

$y = 0$. $\quad dy = 0$.

$x = a$

$dx = 0$

$$\int_{OA} x^2 dx + y^2 dx - 2xy\,dy + \int_{AB} x^2 dx + y^2 dx - 2xy\,dy$$

$$\int_{OA} x^2 dx + y^2 dy - 2xy\,dy \Rightarrow dy = 0 \quad y = 0.$$

$$= \int_0^a x^2 dx + (0)dx - 2x(0)(0) \quad \text{[(0,a) are limits of x]}$$

$$\Rightarrow \left[\frac{x^3}{3}\right]_0^a \Rightarrow \frac{a^3}{3}.$$

$$\int_{AB} x^2 dx + y^2 dx - 2xy\,dy \qquad dx = 0, \; x = a.$$

$$\int_0^b -2ay\,dy \qquad \text{(0,b) are limits of y)}$$

$$-\frac{2ab^2}{2} \Rightarrow$$

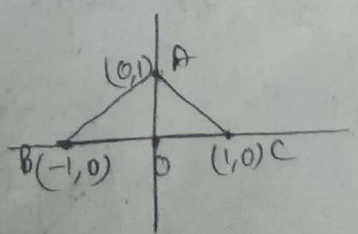
Total work done $\Rightarrow \boxed{\left(\frac{a^3}{3} - ab^2\right) J}$

Ex $W = \int y^2 dx - x^2 dy$ where C is a $\Delta$ with vertices (1,0), (0,1), (−1,0) Find W.D.

$$\int_{AB} y^2 dx - x^2 dy \qquad (y-0) = \frac{1}{1}(x+1)$$

$$y = x+1$$

$$dy = dx.$$




$$\int_{AB} (y^2 - x^2)\,dx$$

$$\because y = (x+1)$$

$$\int_{AB} \left((x+1)^2 - x^2\right) dx \Rightarrow \left[\frac{(x+1)^3}{3}\right]_{-1}^{0} - \left[\frac{x^3}{3}\right]_{-1}^{0} \Rightarrow \boxed{-1/3}$$

$$\int_{BC} y^2dx - x^2dy \qquad \begin{aligned} y&=0 \\ dy&=0. \end{aligned}$$

$$\Rightarrow \int_{BC} y^2dx - x^2dy = 0.$$

$$\int_{AC} y^2dx - x^2dy \qquad \begin{aligned} (y-0) &= \frac{1}{-1}(x-1) \\ y &= 1-x. \\ dy &= -dx. \end{aligned}$$

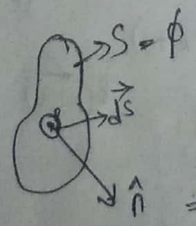
$$\int_1^0 \left((1-x)^2 + x^2\right) dx.$$

$$\left[\frac{(1-x)^3}{3}\right]_0^1 + \left[\frac{x^3}{3}\right]_0^1$$

$$0 + 1/3.$$

$$\text{Total WD} = -1/3 + 1/3 + 0 = 0.$$

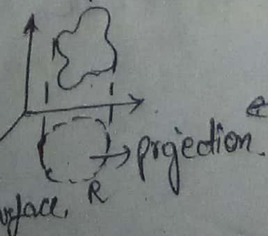
**SURFACE INTEGRAL:** Let S be a smooth surface and let $\hat{n}$ be the unit outward drawn normal to this surface S at any point P having elementary area $\vec{dS}$. Then the $\iint_S \vec{F}\cdot\hat{n}\,\vec{dS}$ over the surface S is known as the surface integral of the vector function F over the surface S.




$$\Rightarrow \frac{\nabla\phi}{|\nabla\phi|}$$

1). If the given surface is projected on the x-y plane, then we call it as projection R

we calculate the integral over the projection and call it the integral over the surface




We take the projection of the 3-D surface S on the X-Y plane because double integral is over a plane

$$\iint \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{\vec{F}\cdot\hat{n}\,dx\,dy}{(\hat{n}\cdot\hat{k})}$$

2) If the projection is in y-z plane.

$$\iint_R \frac{\vec{F}\cdot\hat{n}\,dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

The surface integral is the calculation of flux.

3) If the projection is in the x-z plane.

$$\iint_R \frac{\vec{F}\cdot\hat{n}\,dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

Ex Calculate $\iint_S \vec{F}\cdot\hat{n}\,d\vec{s}$ where $\vec{F} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\,k$ and.

$S: 2x + y + 2z = 6$

$2z = 6 - 2x - y.$

$2x + y + 2z - 6 = 0$

$\hat{n} = \vec{\nabla}S / |\nabla$

$\vec{\nabla}\vec{S} = \frac{\partial S}{\partial x}\hat{i} + \frac{\partial S}{\partial y}\hat{j} + \frac{\partial S}{\partial z}\hat{k}$

$\rightarrow 2\hat{i} + \hat{j} + 2\hat{k}$

$\hat{n} = \frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$

$\vec{F}\cdot\hat{n} = ((x+y^2)\hat{i} - 2x\hat{j} + 2yz\,\hat{k})\cdot\left(\frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})\right)$

$$\iint \frac{1}{3}\left(2(x+y^2) - 2x + 4yz\right)ds$$

in X-Y plane

$\longrightarrow$

$$\iint \frac{\vec{F}\cdot\hat{n}\,dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\frac{1}{3}\int_0^3\int_0^{6-2x}\left(2(x+y^2)-2x+2y(6-2x-y)\right)dx\,dy.$$

~~when~~ in x-y plane z=0.

S: $2x+y=6$.

~~y = 6-2x~~ $y=6-2x \Rightarrow 0-(6-2x)$

$x \Rightarrow 3 \Rightarrow 0-3.$

$$\frac{3}{2}\times\frac{1}{3}\int_0^3 2x(6-2x)+\frac{1}{3}\left[y^3\right]_0^{6-2x}-2x\left[y\right]_0^{6-2x}+\frac{12}{2}\left[y^2\right]_0^{6-2x}-\frac{4x}{2}\left[y^2\right]_0^{6-2x}-\frac{2}{2}\left[y^2\right]_0^{6-2x}\,dx$$

$$\frac{1}{2}\int_0^3 12x-4x^2+\frac{1}{3}(6-2x)^3-2x(6-2x)+6(6-2x)^2-2x(6-2x)^2-(6-2x)^2\,dx$$

(81)

Ex $\vec{F}=yz\hat{i}+zx\hat{j}+xy\hat{k}$ over $x^2+y^2+z^2=1$ : S in the first octave

$$\hat{n}=\frac{\vec{\nabla}S}{|\vec{\nabla}S|}$$

$$\vec{\nabla}S=\frac{\partial S}{\partial x}\hat{i}+\frac{\partial S}{\partial y}\hat{j}+\frac{\partial S}{\partial z}\hat{k}$$

$$\Rightarrow 2x\hat{i}+2y\hat{j}+2z\hat{k}.$$

$$\sqrt{(2x)^2+(2y)^2+(2z)^2}$$

$$2\sqrt{x^2+y^2+z^2}$$

$$\hat{n}\Rightarrow\frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2}$$

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k}).\quad \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$\int_0^{2a}$$

$-3ab(a+b)$

$-3a^2b-3ab^2$

$$\vec{F}\cdot\hat{n}=\frac{3xyz}{\sqrt{x^2+y^2+z^2}}$$

$$\iint \vec{F}\cdot\hat{n}\cdot\vec{dS} \Rightarrow \iint \frac{\vec{F}\,\hat{n}\,dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$x^2+y^2=1$

$$\int_0^1\int_0^{1-x^2} \frac{3xyz}{\sqrt{x^2+y^2+z^2}} \times \frac{\sqrt{x^2+y^2+z^2}}{z}\,dx\,dy$$

$$\int_0^1 \frac{3x}{2}\left[y^2\right]_0^{1-x^2} dx$$

$(x^4-2x^2+1)x$

$$\frac{3}{2}\int_0^1 x(1-x^2)^2\,dx$$

$$\frac{3}{2}\int_0^1 (x^5-2x^3+x)\,dx$$

$$\frac{3}{2}\left(\left[\frac{x^6}{6}\right]_0^1-\left[\frac{x^4}{4}\right]_0^1+\left[\frac{x^2}{2}\right]_0^1\right)$$

$$\frac{3}{2}\left[\frac{1}{6}-\frac{1}{2}+\frac{1}{2}\right]$$

$$\frac{3}{2}\times\frac{1}{6} \Rightarrow \boxed{1/4}$$

$$\frac{3}{2}\int_0^1 x(1-x^2)\,dx$$

$$\frac{3}{2}\int_0^1 (x-x^3)\,dx$$

$$\frac{3}{2}\left(\left[\frac{x^2}{2}\right]_0^1-\left[\frac{x^4}{4}\right]_0^1\right)$$

$$= \left(\frac{1}{2}-\frac{1}{4}\right)\times\frac{3}{2} \Rightarrow \boxed{3/8}$$

$(6-2x)^2\ (6-2x-1) \Rightarrow (6-2x)^2(5-2x)$

$(6-2x)\left(\frac{1}{3}(6-2x)^2-2x\right)$

$$\frac{36+4x^2-24x}{3}$$

$12+\frac{4}{3}x^2-8x-2x$

$6-2x\left(12+\frac{4}{3}x^2-10x\right)$

$$12\left[\frac{x^2}{2}\right]_0^3 - 4\left[\frac{x^3}{3}\right]_0^3 + \frac{1}{3}216[x]_0^3-\frac{18}{3}\left[\frac{x^4}{4}\right]_0^3-\frac{216}{3}\left[\frac{x^2}{2}\right]_0^3-$$

$$\frac{72}{2}\left[\frac{x^3}{3}\right]_0^3-$$

$$\frac{12}{2}\left[x^2\right]_0^3+\frac{4}{3}\left[x^3\right]_0^3+$$

$$216[x]_0^3+\frac{24}{3}\left[x^3\right]_0^3-\frac{144}{2}\left[x^2\right]_0^3$$

$3\times36\times2x$

$\frac{336}{216}$

$\frac{108}{2}$

$\frac{36}{216}$

$(a-b)216x$

$8\times6\times4x^2$

$\frac{24}{72}$

$\frac{24}{144}$

$$\iint \vec{F}\cdot\hat{n}\,d\vec{s} = \iint \frac{\vec{F}\cdot\hat{n}\,dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$y+2z=6$

$$\frac{\cancel{8}}{\cancel{x}}\times\frac{1}{\cancel{8}}\iint(\cancel{2}(x+y^2)-\cancel{2}x+\overset{2}{\cancel{4}}yz)\,dy\,dz$$

$$\iint(x+y^2)-2x+2yz)\,dy\,dz$$

$$\int_0^3\int_0^{6-2z}\left(\frac{6-2z-y}{2}+y^2-6-2z-y+2yz\right)dy\,dz$$

$2y$ $dy$

$$3[y]_0^{6-2z}-z[y]_0^{6-2z}-\frac{1}{4}[y^2]_0^{6-2z}+\frac{1}{3}[y^3]_0^{6-2z}-$$

$$[y]_0^{6-2z}-2z[y]_0^{6-2z}-\frac{1}{2}[y^2]_0^{6-2z}+\frac{2z}{2}[y^2]_0^{6-2z}$$

$\int(a-bx)^3\,dx$

$$\int_0^3\left(\frac{1}{3}[y^3]_0^{6-2z}+[y]_0^{6-2z}(3-z-6-2z)+[y^2]_0^{6-2z}\left(-\frac{1}{4}-\frac{1}{2}+z\right)\right)dz$$

$$\int_0^3\left(\frac{1}{3}(6-2z)^3+(6-2z)^2\left(z-\frac{3}{4}\right)-(6-2z)(3z+3)\right)dz$$

$(6-2z)(36+12z+4z^2)$

$(36+4z^2-24z)\left(z-\frac{3}{4}\right)$

$(6-2z)\left(12+4z+\frac{4}{3}z^2-3z-3\right)$

$36z+4z^3-24z^2-27-3z^2+18z$

$(6-2z)\left(9+z+\frac{4}{3}z^2\right)$

$54+6z+8z^2-18z-2z^2-\frac{8}{3}z^3+36z+4z^3-24z^2-27-3z^2+18z$

$4z^3+42z+\left(21-\frac{8}{3}\right)z^2+27$

$61+21\times9+21\times9-8\times3+27\times3$

$21-\frac{8}{3}$ 54 27 27

$61+21\times9+\left(21-\frac{8}{3}\right)\frac{1}{3}\times 27^9+27\times3$

Green's Theorem in a Plane.

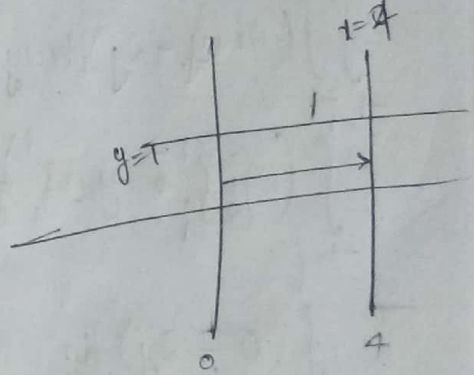
If $f_1(x,y)$ and $f_2(x,y)$ are continuous functions having cont. partial derivatives over a region $R$ bounded by a single closed curve $C$ in the $(x,y)$ plane then

$$\oint_C f_1 dx + f_2 dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy.$$

Using green's theorem evaluate $\oint_C \vec{F}(\vec{r})\,d\vec{r}$ counter clockwise where $\vec{F} = 3x^2\hat{i} - 4xy\hat{j}$ and $C$ is a rectangle bounded by $0 \le x \le 4$ and $0 \le y \le 1$.

$$\vec{F}\cdot d\vec{r} \Rightarrow (3x^2\hat{i} - 4xy\hat{j})(dx\hat{i} + dy\hat{j})$$

$$3x^2dx - 4xydy$$




$$\oint 8x^2dx - 4xy\,dy.$$

Green's theorem

$$\oint_C f_1dx - f_2dy = \iint_R \left(-\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x}\right) dx\,dy.$$

$$\oint 8x^2dx - 4xy\,dy = \iint \left(\frac{\partial}{\partial x}(-4xy) - \frac{\partial}{\partial y}(8x^2)\right) dx\,dy.$$

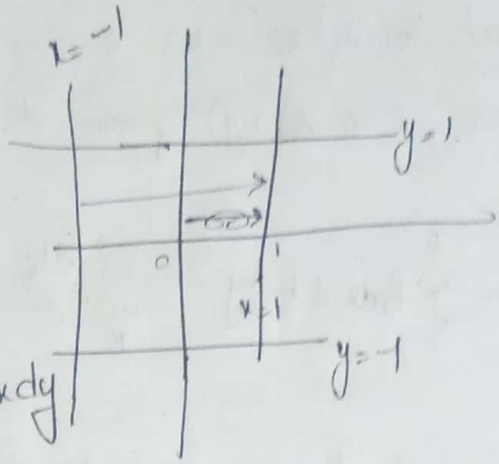
$$\int_0^1\int_0^4 -4y\,dx\,dy.$$

$$\int_0^1 -4y\,[x]_0^4\,dy \Rightarrow \int_0^1 16y\,dy \Rightarrow 16\left[\frac{y^2}{2}\right]_0^1 \Rightarrow -8$$

Ex Use Green's theorem to evaluate $\oint_C (x^2+xy)dx + (x^2+y^2)dy$

over a region bounded by $x = \pm 1$ $y = \pm 1$

$\oint_C (x^2+xy)\,dx + (x^2+y^2)\,dy$




Green's theorem

$$\oint_C f_1 dx + f_2 dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

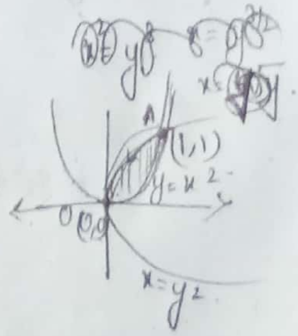
$$\iint_R \left(\frac{\partial}{\partial x}(x^2+xy) - \frac{\partial}{\partial y}(x^2+y^2)\right) dx\,dy$$

$$\int_{-1}^{1}\int_{-1}^{1} (2x+y-2y)\,dx\,dy$$

$$\int_{-1}^{1} \left(2\left[\frac{x^2}{2}\right]_{-1}^{1} - y\,[x]_{-1}^{1}\right) dy$$

$$\int_{-1}^{1} (0 - 2y)\,dy \Rightarrow -2\left[\frac{y^2}{2}\right]_{-1}^{1} \Rightarrow 0.$$

Ex Verify Green's theorem for $\oint x^2y\,dx + xy^2\,dy$ taken along the closed path along $y = \sqrt{x}$, $y = x^2$.




$$\int_{\overrightarrow{OA}} + \int_{\overrightarrow{AO}} \Rightarrow \oint_C.$$

$y = y^4$

$y - y^4 = 0$

$y(1-y^3) = 0$

$y^3 = 1$

$\boxed{y=1}$

$y = x^2$

$dy = 2x\,dx$

$$\int_0^1 x^4 dx + 2x^5\,dx + \int_1^0 2y^5 dy + y^4 dy.$$

$x = y^2$

$dx = 2y\,dy.$

$$\Rightarrow \int_0^1 (x^4 + 2x^6)\,dx + \int_1^0 (2y^6 + y^4)\,dy$$

$$\left[\frac{x^5}{5}\right]_0^1 + 2\left[\frac{x^7}{7}\right]_0^1 + 2\left[\frac{y^7}{7}\right]_1^0 + \left[\frac{y^5}{5}\right]_1^0$$

$$\Rightarrow \frac{1}{5} + \frac{2}{7} - \frac{2}{7} - \frac{1}{5} \Rightarrow \frac{2}{5} + \frac{4}{7} \Rightarrow \cancel{\frac{}{}} \Rightarrow \cancel{\frac{}{}} \quad 0.$$

By Green's theorem,

$$\oint F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy.$$

$$\oint x^2 y\,dx + xy^2 dy = \iint_R \left(\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(x^2 y)\right) dx\,dy.$$

$$\Rightarrow \int_0^1 \int_{x^2}^{\sqrt{x}} (y^2 - x^2)\,dx\,dy$$

$$\int_0^1 \left(\left[\frac{y^3}{3}\right]_{x^2}^{\sqrt{x}} - x^2 [y]_{x^2}^{\sqrt{x}}\right) dx \qquad \left[\frac{y^3}{3}\right]_{\sqrt{x}}^{x^2}$$

$$\int_0^1 \frac{x\sqrt{x} - x^6}{3} - x^2(\sqrt{x} - x^2)\,dx \qquad \frac{x^6 - x\sqrt{x}}{3} -$$

$$\int_0^1 \left(\frac{x^{3/2}}{3} - \frac{x^6}{3} - x^{5/2} + x^4\right) dx \qquad x^2(x^2 - \sqrt{x})$$

$2 + 1/2$

$$\frac{2}{5} \times \frac{\left[x^{5/2}\right]_0^1}{3} - \frac{\left[x^7\right]_0^1}{7\times 3} - \frac{2}{7}\left[x^{7/2}\right]_0^1 + \frac{\left[x^5\right]_0^1}{5} \qquad \frac{x^6}{3} - \frac{x\sqrt{x}}{3} - x^4 + x^2\sqrt{x}.$$

$\frac{3}{2}+1$

$\frac{5}{2}+1$

$$\frac{2}{15} - \frac{1}{21} - \frac{2}{7} + \frac{1}{5}$$

7 | 5, 21, 7, 5
5 | 5, 3, 1, 5
3 | 3, 3, 1, 1
  | 1

$$\frac{14 - 5 - 30 + 21}{105} \qquad \boxed{0}$$

135 ÷ 3 = 105

$\frac{14}{7}$

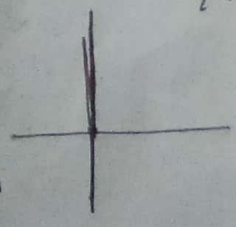
21
14
35

25

$y^2(y-1) = 0$

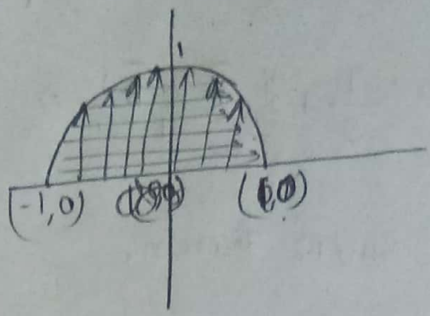
$y = 1 \quad x = 1$

$x = y^{3/2} \quad y \times \sqrt{y} \quad x^2 = y^3 \quad x^2 = y^3 \quad y^2 = y \quad x = y$



Use green's theorem to evaluate $\oint_C (2x^2-y^2)dx + (x^2+y^2)dy$ where C is a boundary in the x-y plane of the area bounded by x-axis and $x^2+y^2=1$ in the upper half plane

$$\oint_C (2x^2-y^2)dx + (x^2+y^2)dy.$$




By Green's Theorem.

$$\oint_C f_1 dx + f_2 dy = \iint_R \left(\frac{\partial}{\partial x} f_2 - \frac{\partial}{\partial y} f_1\right) dx\,dy.$$

$$\oint_C (2x^2-y^2)dx + (x^2+y^2)dy = \iint_R \left(\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)\right) dx\,dy$$

$$\iint_R (2x+2y)\,dx\,dy \Rightarrow 2\iint_R (x+y)\,dx\,dy.$$

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x+y)\,dx\,dy.$$

$$\int_{-1}^{1}\left( x[y]_0^{\sqrt{1-x^2}} + \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} \right) dx$$

$$\frac{-1}{2}\int_{-1}^{1} -2x\sqrt{1-x^2}\,dx + \int_{-1}^{1}\frac{1-x^2}{2}\,dx$$

$1-x^2 = t$

$-2x\,dx = dt.$

$$\frac{-1}{2}\int \sqrt{t}\,dt$$

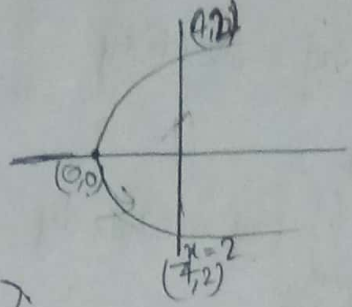
$$\cancel{\frac{-1}{2}\times\frac{3}{2}(t)^{3/2}} + \frac{[x]_{-1}^{1}}{2} - \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$\left(\left[1-x^2\right]^{3/2}\right)_{-1}^{1}$$

$$1 - \frac{1}{3} \Rightarrow \frac{2}{3}$$

Ex Verify Green's Theorem.

$\oint_C (x^2-2xy)dx + (x^2y+3)dy$ over a region bounded by $y^2=8x$ and $x=2$.




$$\left\{\int_{y^2=8x} (x^2-2xy)dx + (x^2y+3)dy\right\} +$$

$$\left[\int_{x=2} (x^2-2xy)\,dx + (x^2y+3)\,dy\right]$$

$2y\,dy = 8\,dx$

$dx = y/4\,dy$.

$$\left[\int_4^{-4} \left(\frac{y^4}{64} - \frac{2y^3}{4}\right)\frac{y}{4}dy + \left(\frac{y^5}{64}+3\right)dy\right] +$$

$$\int_{-4}^{4} (4y+3)dy.$$

$\begin{array}{r}164\\ 4\\ \hline 256\end{array}$

$$\int_4^{-4} \left(\frac{y^5}{256}dy - \frac{y^4}{16}dy + \frac{y^5}{64} + 3\,dy\right) + \int_{-4}^{4} (4y+3)dy.$$

$$\cancel{\left[\frac{y^6}{6\times 256}\right]_4^{-4}} - \left[\frac{y^5}{16\times 5}\right]_4^{-4} + \cancel{\left[\frac{y^6}{6\times 64}\right]_4^{-4}} + 3[y]_4^{-4} + 4\cancel{\left[\frac{y^2}{2}\right]_{-4}^{4}} + 3[y]_{-4}^{4}$$

$$+ \frac{2\times(4)^5}{16\times 5} - 24 + 24$$

$$\Rightarrow \frac{4\times 4\times 4\times 4\times 4}{8\times 5} \rightarrow \boxed{\frac{128}{5}}$$

- Green's Theorem

$$\oint_C F_1dx + F_2dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy.$$

$$\oint_C (x^2-2xy)\,dx + (x^2y+3)\,dy = \iint_R \left(\frac{\partial}{\partial x}(x^2y+3) - \frac{\partial}{\partial y}(x^2-2xy)\right) dx\,dy$$

$$\iint_R (2xy - 2x)\,dx\,dy$$

$$2\int_{-4}^{4}\int_{y^2/8}^{2} (xy - x)\,dx\,dy$$

$$2\int_{-4}^{4}\left( y\left[\frac{x^2}{2}\right]_{y^2/8}^{2} - \left[\frac{x^2}{2}\right]_{y^2/8}^{2} \right) dy$$

$$2\int_{-4}^{4} y\left(2 - \frac{y^4}{128}\right) - \left(2 - \frac{y^4}{128}\right) dy$$

$$\cancel{2\left[\frac{y^2}{2}\right]_{-4}^{4}} - \cancel{\frac{1}{128}\left[\frac{y^6}{6}\right]_{-4}^{4}} - 2[y]_{-4}^{4} + \frac{1}{128}\left[\frac{y^5}{5}\right]_{-4}^{4}$$

$$2\left(-16 + \frac{2\times4\times4\times4\times4\times4}{128\times5}\right)$$

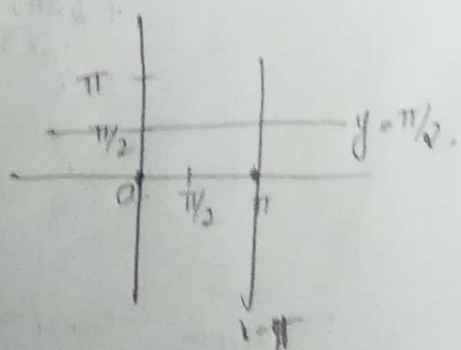
$$2\left(\frac{16-60}{5}\right) \to -\frac{88}{5}$$

$32 \quad \frac{^2 4\times4\times4\times4\times4}{\cancel{170}\ 85}$ $\quad 16\times5 \to 60$ $\quad 128\times2$

$\begin{array}{r} 85 \\ 32 \\ \hline 170 \\ 2550 \\ \hline 2720 \\ 256 \\ \hline 2464 \end{array} \qquad \begin{array}{r} 60 \\ 16 \\ \hline 44 \end{array} \qquad \frac{\cancel{2464}}{\cancel{85}}$

Eg $\oint_C e^{-x}(\sin y\,dx + \cos y\,dy)$ C is a rectangle with vertices

$(0,0)\ ,\ (\pi,0)\ ,\ (\pi,\pi/2)\ ,\ (0,\pi/2)$




$$\iint \left( \frac{\partial}{\partial x}(e^{-x}\cos y) + \frac{\partial}{\partial y}(e^{-x}\sin y) \right) dx\,dy$$

$$\iint \left(-e^{-x}\cos y - e^{-x}\cos y\right) dx\,dy$$

$$-2\iint e^{-x}\cos y\,dx\,dy$$

## Gauss - Divergence Theorem:

The surface integral of Normal component of a vector point function $\vec{F}$ over ove closed surface is equal to volume integral of the Divergence of $\vec{F}$ taken throughout the volume V enclosed by the surface S. This is known as gauss divergence theorem.

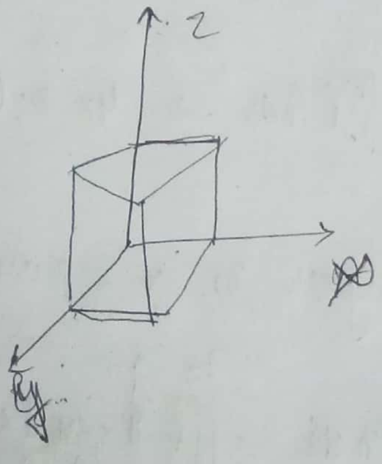
$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (\nabla\cdot\vec{F})\, dv.$$

Eg Verify Gauss Divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

S. cube $x=0 \cdot x=2.$

$y=0 \; y=2$

$z=0 \; x=2.$




Gauss Divergence theorem:

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (\nabla\cdot\vec{F})\, dv.$$

RHS

$$\int_0^2\int_0^2\int_0^2 (4z - y)\, dx\, dy\, dz$$

$$\int_0^2\int_0^2 (8z - 2y)\, dy$$

$$\int_0^2 (16z - 4)\, dz$$

$$= 32 - 8 \Rightarrow 24 //.$$

RHS

On face OABC (x-y plane).

$\iint \vec{F}\cdot\hat{n}\,ds$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint \vec{F}\cdot\hat{k}\,\frac{dx\,dy}{(\hat{n}\cdot\hat{k})}$$

Because the perpendicular to a surface in x-y plane is $\hat{k}$. following ~~we~~ the formula of surface integral.

$$\iint_S \vec{F}\cdot -\hat{k} \times \frac{dx\,dy}{|-\hat{k}\cdot\hat{k}|}$$

$$\int_0^2\int_0^2 -yz\,dx\,dy \Rightarrow \int_0^2 2yz\,dy \Rightarrow -4z \quad (z=0)$$

$$\Rightarrow 0$$

For the surface ∥ to it but raised on a platform,

$$\iint \vec{F}\cdot\hat{n}\,ds \Rightarrow 4z \Rightarrow (z=2) \Rightarrow 8$$

For the surfaces in x-z plane.

$$\iint_S \vec{F}\cdot\hat{n}\,ds \Rightarrow \int_0^2\int_0^2 \vec{F}\cdot -\hat{j} \times \frac{dx\,dz}{|-\hat{j}\cdot\hat{j}|} \Rightarrow \int_0^2\int_0^2 y^2\,dx\,dz$$

$$\Rightarrow \int_0^2 2y^2\,dz \Rightarrow 4y^2 = 0$$

for the surface in x-z plane

for the surface in plane x-z with y=2.

$$-4y^2 \Rightarrow -4\times 4 \Rightarrow -16$$

for the surface in y-z plane

$$\iint_S \vec{F}\cdot\hat{n}\,ds \Rightarrow \iint \vec{F}\cdot -\hat{i} \times \frac{dy\,dz}{|-\hat{i}\times\hat{i}|} \Rightarrow \int_0^2\int_0^2 -4xz\,dy\,dz$$

$$= -8xz \Rightarrow -16x = 0.$$

For the surface where $x=2$.

$$\int_0^2\int_0^2 4xz\, dy\, dz.$$

$$\int_0^2 8xz\, dz \Rightarrow 16x \Rightarrow 16\times 2 = 32.$$

$$32-16+8 \Rightarrow 24$$

Ex Apply the Gauss divergence theorem to $\vec{F} = 4x^3 y\hat{i} - x^3 y\hat{j} + x^2 z\hat{k}$ over the and find $\iint_S \vec{F}\cdot\hat{n}\, ds$ over the surface $S: x^2+y^2=a; z=0 \text{ to } z=b$

Gauss Divergence Theorem:

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (\nabla\cdot\vec{F})\, dv.$$

$$\therefore \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^b (12x^2 - x^3 + x^2)\, dx\, dy\, dz.$$

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^b (13x^2 - x^3)\, dx\, dy\, dz$$

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (13bx^2 - bx^3)\, dx\, dy$$

$$x^2\left(\sqrt{a^2-x^2} - x\sqrt{a^2-x^2}\right)$$

$$\left(x^2\sqrt{a^2-x^2}\right)(1-x)$$

$$2.b\int_{-a}^{a} 13x^2\left(\sqrt{a^2-x^2}\right) - x^3\left(\sqrt{a^2-x^2}\right)\, dx$$

$$13\left(x^2\left(\frac{a}{2}\sqrt{a^2-x^2} - \frac{a^2}{2}\right.\right.$$

$$\int_{-a}^{a} \left(x^2\sqrt{a^2-x^2}\right)(1-x)$$

$$(1-x)\int u + \int (\cancel{-1}) \int u$$

$\sqrt{a^2-x^2} = t$

$x^2 = a^2 - t^2$

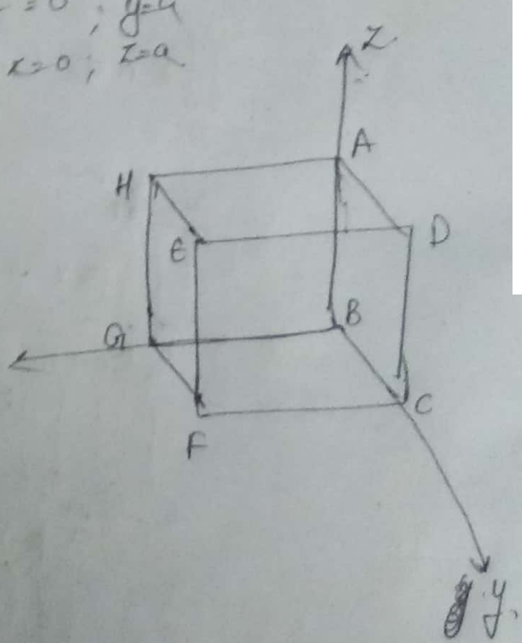
**Ex** Verify Gauss-Divergence theorem ~~over~~ for $\vec{F} = (x^3 - yz)\hat{i} - 2x^2 y \hat{j} + 2\hat{k}$

over the cuboidal surface where $x = 0$ ; $x = a$

$y = 0$ ; $y = a$

$z = 0$ ; $z = a$

Gauss-Divergence theorem:




$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (\nabla\cdot\vec{F})\, dv.$$

$$\int_0^a\int_0^a\int_0^a (3x^2 - 2x^2)\, dx\, dy\, dz$$

$$\int_0^a\int_0^a\int_0^a x^2\, dx\, dy\, dz$$

$$\int_0^a\int_0^a a x^2\, dy\, dz \Rightarrow \int_0^a a^2 x^2\, dx = a^2 \times \frac{a^3}{3} \Rightarrow \boxed{a^5/3}.$$

Surface integral:

For face OBCF.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint (\vec{F}\cdot -\hat{k}) \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = 0.$$

For the face ADEH ⇒ 0.

For the face ABCD ⇒

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a (\vec{F}\cdot -\hat{k}) \frac{dy\,dz}{|-\hat{i}\cdot\hat{k}|} \Rightarrow \int_0^a\int_0^a -3x^2\,dy\,dz.$$

$$-\int_0^a 3ax^2\,dy \Rightarrow -3a^2x^2$$

$(\because x=0)$.

$\Rightarrow 0$ //

For the face GHEF.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint (\vec{F}\cdot\hat{i}) \frac{dy\,dz}{|\hat{i}\cdot\hat{i}|} \Rightarrow \int_0^a\int_0^a 3x^2\,dy\,dz \Rightarrow 3a^2x^2$$

$(x=a)$

$3a^4$ //.

For ~~GBEF~~ ABGH ⇒

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint (\vec{F}\cdot\hat{j}) \frac{dx\,dy}{|-\hat{j}\cdot\hat{j}|} \Rightarrow$$

Theorem:

The line integral tangential form of vector form 'F' taken around the curve the surface integral of the curl of vector function 'F' taken over any open circuit 'S' having 'C' as the component is called Stoke's law

$\cancel{\oint \vec{F}\cdot dr}$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\,\hat{n}\,dS$$

Q using Stoke's theorem evaluate this

$$\oint_C (2x-y)\,dx - yz^2\,dy - y^2 z\,dz$$ where C is a circle $x^2+y^2=1$.

$$\oint (2x-y)\,dx - yz^2\,dy - y^2 z\,dz = \oint F\cdot dr.$$

$$F = (2x-y)\,i - yz^2\,j - y^2 z\,k$$

$$dr = dx\,i + dy\,j + dz\,k.$$

$$\oint_C \vec{F}\cdot dr = \iint_S (\vec{\nabla}\times\vec{F})\,\hat{n}\,ds.$$

$$= \iint \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & (-yz^2) & (-y^2 z) \end{vmatrix}$$

$$= \iint i(-2y+2zy) - j(0-0) + k(0+1)$$

$$= \iint (-2y+2zy)\,i + k.$$

$$= \iint \hat{k}\,\hat{n}\,\frac{dx\,dy}{|\hat{n}\,\hat{k}|}$$

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$$= 4\int_0^1\int_0^{\sqrt{1-x^2}} dx\,dy$$

$$= \pi a^2$$

Q2 Verify Stokes theorem where the curve 'C' is a square in the plan $z=0$ and $x=0$, $x=a$, $y=0$, $y=a$.

$$F = x^2\hat{i} - xy\hat{j}$$

$$\oint F\cdot dr = \iint (\nabla\times F)\,\hat{n}\,ds$$

$$R.H.S = \iint \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix} \cdot n\,ds$$

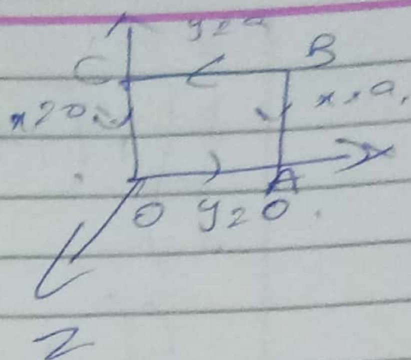
$$= \iint i(0) - j(0-0) + k(-y)$$

$$= \iint -ky$$

$$= \iint -y\hat{k}\,\hat{n}\,\frac{dx\,dy}{|\hat{n}\hat{k}|}$$

$$= \int_0^a\int_0^a -y\,dx\,dy \qquad \left.-\frac{y^2}{2}\right]_0^a$$

$$= -\frac{a^3}{2} \qquad -\frac{a^2}{2}a$$




OA $\int x^2\,dx - xy\,dy$

$y = 0 \Rightarrow dy = 0.$

$= \int_0^a x^2\,dx = \frac{x^3}{3}$

$= a^3/3.$

AB
$x = a$
$dx = 0$

$\int_0^a -ay\,dy = -a^3/2.$

BC
$=$

$\int_a^0 x^2\,dx = -a^3/3.$

CO
$=$

$\int x^2\,dx - xy\,dy = 0$ //
$x = 0 \quad dx = 0$

Hence area $= \frac{a^3}{3} - \frac{a^3}{2} \cancel{=} \frac{a^3}{3}$

$= -a^3/2$ //.

Q. $\vec{F} = (2y+z)\,i + (x-z)\,j + (y-x)\,\hat{k}$. Verify Stokes theorem for this function F. Taken portion on the curve.

$F \cdot dr = (2y+z)i + (x-z)i + (y-z)k \cdot (dx\,i + dy\,j + dz\,k)$

$$\iint \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y+z & x-z & y-x \end{vmatrix} \cdot n\,ds.$$

$$\iint i(1+1) - j(-1-1) + k(1\cancel{+}2)\ \hat{n}\cdot ds$$

$$\iint 2i + 2j \cancel{+} \cancel{3}k\ \hat{n}\, ds.$$

$$\iint 2i + 2j + \cancel{2k} \cancel{+} k\ \hat{n}\ ds.$$

$$\cancel{\iint 2(i+j+k) + k\ \hat{n}\ ds.}$$

$$\cancel{\iint 2(i+j+k) + k\ \hat{n}\ ds} = \iint 2(i+j+k)$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial y} k\right) S.$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{i+j+k}{\sqrt{3}}.$$

$$= \int_0^1 \int_0^{1-x} 2i + 2j \cancel{+} \cancel{3}k \cdot \frac{(i+j+k)}{\sqrt{3}} \frac{dx\,dy}{1/\sqrt{3}}$$

$$= \int_0^1 \int_0^{1-x} (2i+2j-k)(i+j+k)\, dx\, dy.$$

$$= \int_0^1 \int_0^{1-x} 2 + 2 - 1$$

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$$\int_0^1\int_0^{1-x} 3\,dx\,dy$$

$$\int_0^1 3(1-x)\,dx$$

$$\int_0^1 3-3x\,dx$$

$$\left.3x-\frac{3x^2}{2}\right]_0^1$$

$$3-\frac{3}{2}=\frac{3}{2}\;//$$

L.H.S

$\oint F\cdot dr = \oint$ .